Question

(a) Let

$$\begin{aligned} \mathbf{a} &= 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \\ \mathbf{b} &= \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \\ \mathbf{c} &= -\mathbf{i} + \mathbf{j} - 3\mathbf{k} \end{aligned}$$

Evaluate the following:

$$\mathbf{a}\cdot\mathbf{b},\quad \mathbf{a}\cdot\mathbf{c},\quad \mathbf{b}\times\mathbf{c},\quad \mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$$

$$\mathbf{c}\cdot\mathbf{b}\times\mathbf{a},\quad \mathbf{c}\times(\mathbf{b}\times\mathbf{c}),\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$

Are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, linearly independent?

(b) Find the equations of the two planes which contain the line

$$x - 5 = \frac{y-1}{-1} = \frac{z+3}{3}$$

and which make an angle of 60° witht the plane $y - z = 0$.

Answer

(a)

$$\begin{aligned} \mathbf{a} &= 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \\ \mathbf{b} &= \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \\ \mathbf{c} &= -\mathbf{i} + \mathbf{j} - 3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 12 \\ \mathbf{a}\cdot\mathbf{c} &= -8 \\ \mathbf{b}\times\mathbf{c} &= (2,-1,-1) \\ \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= 6 \\ \mathbf{c}\cdot\mathbf{b}\times\mathbf{a} = -\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= -6 \\ \mathbf{c}\times(\mathbf{b}\times\mathbf{c}) &= (-4,-7,-1) \\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (4,4,4) \end{aligned}$$

Since $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq 0$ $\quad\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, are independent.

(b) Suppose the plane has equation $ax + by + cz = k$

Then $(a, b, c) \cdot (1, -1, 3) = 0$

So $a - b + 3c = 0$

Also $5a + b - 3c = k$ So $k = 6a$

Then $(a, b, c) \cdot (0, 1, -1) = b - c$

So $b - c = \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{2}\frac{1}{2}$

So $2b^2 - 4bc + 2c^2 = a^2 + b^2 + c^2$

i.e. $b^2 + c^2 - a^2 - 4bc = 0$

But $a = b - 3c$

giving $2c(b - 4c) = 0$ So $c = 0$ or $b = 4c$

If $b = 4c$ then $a = c$ and $b = 4a$

Giving

$$x + 4y + z = 6$$

If $c = 0$ then $a = b$

Giving

$$x + y = 6$$