Question

(i) Show that if

$$I_n = \int_0^\infty x^n e^{-x}\, dx, \quad n \geq -\frac{1}{2},$$

then

$$I_n = nI_{n-1}$$

Hint: you may assume that $\lim_{x\to+\infty}(x^p e^{-x}) = 0$ for any value of p.

(ii) Evaluate I_0 and hence calculate I_8.

(iii) Given that when $n = -\frac{1}{2}$, $I_{-\frac{1}{2}} = \sqrt{\pi}$, calculate $I_{\frac{5}{2}}$.

Answer

(i) $I_n = \int_0^\infty x^n e^{-x}\, dx \quad n \geq 0$

$$\begin{array}{ll} u = x^n & \dfrac{dv}{dx} = e^{-x} \\ du = nx^{n-1} & v = -e^{-x} \end{array}$$

Integrate by parts:

$$\begin{aligned} I_n &= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}\, dx \\ &= \lim_{x\to\infty}\left[-x^n e^{-x}\right] + 0 + n\int_0^\infty x^{n-1}e^{-x}\, dx \\ &= n\int_0^\infty x^{n-1}e^{-x}\, dx \end{aligned}$$

by hint

$\Rightarrow \underline{I_n = nI_{n-i}}$

(ii) $I_0 = \int_0^\infty x^0 e^{-x}\, dx = \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = +1$

Therefore

$$\begin{aligned} I_8 = 8I_7 = 8\times 7I_6 = \cdots &= 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1\times 1 \\ &= 8! = 40320 \end{aligned}$$

(iii) If $I_{-\frac{1}{2}} = \sqrt{\pi}$,

$$I_{\frac{5}{2}} = \frac{5}{2} I_{\frac{3}{2}} = \frac{5}{2} \times \frac{3}{2} I_{\frac{1}{2}} = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} I_{-\frac{1}{2}} = \frac{15}{8}\sqrt{\pi}$$