QUESTION

Consider the matrices

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(i) Compute the following matrices and their determinants:

$$A + B, AB, BA.$$

(ii) Find the eigenvalues and the corresponding normalised eigenvectors for the matrix A.

(iii) Write down the quadratic form associated with A and express it in diagonal form. Say giving brief reasons for your answer, whether or not this quadratic form represents an ellipsoid.

ANSWER

(i)

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 1 & 4 & 0 \\ 4 & 1 & 5 \\ 0 & 5 & 1 \end{pmatrix}, \ AB = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}, \ BA = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 7 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

$\det(A + B) = -40, \ \det(AB) = \det(BA) = 0$

(ii) The eigenvalues of A satisfy $(1-\lambda)\left[(1-\lambda)(1-\lambda) - 16\right] - 3\left[3(1-\lambda) - 0\right] + 0 = 0$ which factorises as $(1-\lambda)\left[(1-\lambda)^2 - 25\right] = 0$ so $\lambda = 1$ or $1 - \lambda = \pm 5$ with roots $\lambda = -4, 1, 6$.

For the eigenvalues:

$\lambda = -4$ gives $\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $5x = -3y$ and $5z = -4y$.

This gives a normalised eigenvector $\frac{1}{\sqrt{50}} \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix}$

$\lambda = 1$ gives $\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $y = 0$ and $3x = -4z$.

This gives a normalised eigenvector $\frac{1}{5} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

$\lambda = -4$ gives $\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $5x = 3y$ and $5z = 4y$.

This gives a normalised eigenvector $\frac{1}{\sqrt{50}} \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

(iii) The quadratic form associated to A is $x^2 + y^2 + z^2 + 6xy + 8yz$ which does not represent an ellipsoid since it diagonalises to $X^2 - 4Y^2 + 6Z^2$ which has both negative and positive coefficients.