Question

The height of the ground in kilometers near an extinct volcano is given by the formula :

$$h = \exp\left(-(x^2 + y^2 - 0.25)^2\right)$$

where x and y are the distances in kilometers from the centre of the crater in the north and east directions respectively.

Let $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Derive a formula for $\dfrac{\partial h}{\partial \theta}$, and show that $\dfrac{\partial h}{\partial \theta} = 0$. What is the physical meaning of this result?

(b) Find a general formula for $\dfrac{\partial h}{\partial r}$, and show that $\dfrac{\partial h}{\partial r} = 0$ for $r = 0$ and $r = 0.5$. What is the physical meaning of this result?

Answer

(a) $x = r\cos\theta \Rightarrow \dfrac{\partial x}{\partial \theta} = -r\sin\theta = -y$

$y = r\sin\theta \Rightarrow \dfrac{\partial y}{\partial \theta} = r\cos\theta = x$

Chain rule $\Rightarrow \dfrac{\partial h}{\partial \theta} = \dfrac{\partial h}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial h}{\partial y}\dfrac{\partial y}{\partial \theta}$

with

$$\begin{aligned}
h &= \exp\left(-(x^2 + y^2 - 0.25)^2\right) \\
\frac{\partial h}{\partial x} &= -4x(x^2 + y^2 - 0.25)\exp\left(-(x^2 + y^2 - 0.25)^2\right) \\
\frac{\partial h}{\partial y} &= -4y(x^2 + y^2 - 0.25)\exp\left(-(x^2 + y^2 - 0.25)^2\right)
\end{aligned}$$

substituting :

$$\begin{aligned}
\frac{\partial h}{\partial \theta} &= -4(x^2 + y^2 - 0.25)\exp\left(-(x^2 + y^2 - 0.25)^2\right) \times (-xy + xy) \\
&\equiv 0
\end{aligned}$$

Physical meaning: (r, θ) are polar coordinates.

We can rewrite $h = \exp\left(-(r^2 - 0.25)^2\right)$. This is independent of the angle θ, so we expect the height to be independent of the angle θ, and the derivative with respect to θ to be zero.

(b) Chain rule $\Rightarrow \dfrac{\partial h}{\partial r} = \dfrac{\partial h}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial h}{\partial y}\dfrac{\partial y}{\partial r}$

$x = r\cos\theta \Rightarrow \dfrac{\partial x}{\partial r} = \cos\theta = \dfrac{x}{r}$

$y = r\sin\theta \Rightarrow \dfrac{\partial y}{\partial r} = \sin\theta = \dfrac{y}{r}$

Substituting:

$$\begin{aligned}\frac{\partial h}{\partial r} &= -4(x^2+y^2-0.25)\exp\left(-(x^2+y^2-0.25)^2\right) \times \left(x\frac{x}{r} + y\frac{y}{r}\right) \\ &= \frac{-4(x^2+y^2)}{r}(x^2+y^2-0.25)\exp\left(-(x^2+y^2-0.25)^2\right)\end{aligned}$$

Now $x = r\cos\theta, y = r\sin\theta \Rightarrow x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

Hence

$$\frac{\partial h}{\partial r} = -4r(r^2-0.25)\exp\left(-(r^2-0.25)^2\right)$$

From the formula $\dfrac{\partial h}{\partial r} = 0$ if $r = 0$ or $r^2 = 0.25 \Rightarrow r = 0.5$

The height has a minimum at the centre of crater $(r = 0)$

The height has maxima at all points on the rim of the crater $(r = 0.5)$