QUESTION

(a) Consider the linear programming problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & x_j \geq 0 & j = 1, \ldots, n \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i & i = 1, \ldots, m, \end{array}$$

where the constraint matrix $A = (a_{ij})$ has rank m, and $m < n$. Prove that a basic feasible solution is an extreme point of the convex set of feasible solutions.

(b) Give a brief explanation of the term *cycling* in the simplex method, and describe two methods by which cycling can be avoided.

(c) Explain what is meant by *soft* constraints, and indicate how they are incorporated into a linear programming problem.

(d) Find and evaluate all basic feasible solutions of the linear programming problem

$$\begin{array}{ll} \text{maximize} & z = 5x_1 + 7x_2 + 4x_3 + 9x_4 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \\ & x_1 + x_2 + 2x_3 - 5x_4 = 4 \\ & 3x_1 + 5x_2 - 2x_3 + x_4 = 8. \end{array}$$

Verify that if the vector (x_1, x_2, x_3, x_4) is a feasible solution, then the vector $(x_1 + 1, x_2, x_3 + 2, x_4 + 1)$ is also a feasible solution. What can you conclude about the problem?

ANSWER

(a) Assume that the variables of the LP problem are $\mathbf{x}$ and the constraints are $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{x} = \begin{pmatrix} \mathbf{x}^B \\ \mathbf{x}^N \end{pmatrix}$ $A = (A^B\ A^N)$, so that constraints are

$$A^B \mathbf{x}^B + A^N \mathbf{x}^N = \mathbf{b}$$

Let $\mathbf{x}^B = A_B^{-1}(\mathbf{b} - A^N \mathbf{x}^N)$ define a basic feasible solution $\mathbf{x}^B = A_B^{-1}\mathbf{b}\ \mathbf{x}_N = \mathbf{0}$. Suppose

$$\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \qquad (1)$$

for feasible solutions $\mathbf{x}_1, \mathbf{x}_2$, where $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < \lambda < 1$.

Let $\mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_1^B \\ \mathbf{x}_1^N \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_2^B \\ \mathbf{x}_2^N \end{pmatrix}$. From (1), $\mathbf{x}_1^N = \mathbf{x}_2^N = \mathbf{0}$, since otherwise $\mathbf{x}^N \neq \mathbf{0}$. For feasibility,

$$\begin{array}{lll} A^B\mathbf{x}_1^B = \mathbf{b} & \text{giving} & \mathbf{x}_1^B = (A^B)^{-1}\mathbf{b} \\ A^B\mathbf{x}_2^B = \mathbf{b} & \text{giving} & \mathbf{x}_2^B = (A^B)^{-1}\mathbf{b} \end{array}$$

Thus $\mathbf{x}_1^B = \mathbf{x}_2^B$ to give $\mathbf{x}_1 = \mathbf{x}_2$, which contradicts the initial assumption $\mathbf{x}_1 \neq \mathbf{x}_2$.

(b) Cycling occurs when a previously generated tableau reappears at some later iteration. To avoid cycling

- perturbation: add the vector $(\epsilon, \epsilon^2, \ldots, \epsilon^m)$ for small positive ϵ to the right hand sides to avoid degeneracy.
- Bland's smallest subscript rule: the pivot column is the one with a nagative x coefficient which has the smallest subscript on the variable; for equal smallest ratios, choose a row containing a basic variable with the smallest subscript.

(c) Soft constraints are ones which should preferably be satisfied, but can be varied at a cost. A soft constraint

$$a_{i1}x_1 + \ldots a_{in}x_n = b_i$$

is replaced by

$$a_{i1}x_1 + \ldots + a_{in}x_n + u_i - v_i = b_i \; u_i \geq 0, \; v_i \geq 0$$

and a contribution $\alpha_o u_i + \beta_i v_i$ for suitable chosen α_i, β_i is added to the cost in the objective function.

(d)

Basic	Value	Feasible	z
x_1, x_2	$6, -2$	No	
x_1, x_3	$3, \frac{1}{2}$	Yes	17
x_1, x_4	$\frac{11}{4}, -\frac{1}{4}$	No	
x_12, x_3	2,1	Yes	18
x_2, x_4	$\frac{22}{13}, -\frac{6}{13}$	No	
x_3, x_4	$-\frac{11}{2}, -3$	No	

$$
\begin{aligned}
(x_1+1) &+ x_2+2(x_3+2)-5(x_4+1) \\
&= x_1+x_2+2x_3-5x_4=4 \\
3(x_1+1) &+ 5x_2-2(x_3+2)+(x_4+1) \\
&= 3x_1+5x_2-2x_3+x_4=8
\end{aligned}
$$

The objective function value of the solution is

$$s(x_1+x)+7x_2+4(x_3+2)+9(x_4+1)=5x_1+7x_2+4x_3+9x_4+22$$

Since any feasible solution can be increased by 22 without affecting feasibility, the problem is unbounded.