QUESTION

Show that

$$\left(\frac{y^2+1}{2y}\right)^2=\frac{y^2}{4}+\frac{1}{2}+\frac{1}{4y^2}.$$

A particle is projected along the positive y- axis so that, initially, at time $t=0$, its position is $y=1$ and its velocity is $\frac{dy}{dt}=1$. The equation of motion of the particle is described by

$$\frac{d^2y}{dt^2}=\frac{y}{4}-\frac{1}{4y^3}.$$

Find the following:

(i) an expression for the velocity as a function of position;

(ii) an expression for the position as a function of time;

(iii) an expression for the velocity as a function of time.

ANSWER

$$\left(\frac{y^2+1}{2y}\right)^2=\frac{y^4+2y^2+1}{4y^2}=\frac{y^2}{4}+\frac{1}{2}+\frac{1}{4y^2}$$

(i) Let $v=\frac{dy}{dt}$. Then $\frac{d^2y}{dt^2}=\frac{dv}{dt}$ and using the chain rule: $\frac{dv}{dt}=\frac{dv}{dy}\frac{dy}{dt}=\frac{dv}{dy}v$. Hence the equation of motion becomes: $\frac{dv}{dy}v=\frac{y}{4}-\frac{1}{4y^3}$ which is a first order separable differential equation. Integrating gives:

$$\begin{aligned}\int v\,dv &= \int \frac{y}{4}-\frac{1}{4y^3}\,dy+C\\ \frac{1}{2}v^2 &= \frac{y^2}{8}+\frac{1}{8y^2}+C\end{aligned}$$

Using initial condition $y=1$ and $v=1$ when

$$\frac{1}{2}=\frac{1}{8}+\frac{1}{8}+C\Rightarrow C=\frac{1}{4}$$

So $\frac{1}{2}v^2=\frac{y^2}{8}+\frac{1}{8y^2}+\frac{1}{4}$ or $v^2=\frac{y^2}{4}+\frac{1}{4y^2}+\frac{1}{2}$

By first part of question : $v^2=\left(\frac{y^2+1}{2y}\right)^2$ and so velocity as a function of position: $\left(\frac{dy}{dt}=\right)v=\frac{y^2+1}{2y}$ (choose positive square root to agree with initial condition $v=1$ when $y=1$)

(ii) Since $v = \frac{dy}{dt}$ we have $\frac{dy}{dt} = \frac{y^2+1}{2y}$ which is first order separable. Integrating gives

$$\int \frac{2y}{y^2+1}\,dy = \int dt + k$$

Either substitute $u = y^2 + 1$ or note that $\int \frac{f'(y)}{f(y)}\,dy = \ln|f(y)|$ gives

$$\ln|y^2+1| = t + K$$

Using initial condition $y = 1$ when $t = 0$ gives

$$\ln 2 = 0 + k \Rightarrow k = \ln 2$$

so $\ln(y^2+1) = t + \ln 2$

Taking exponentials: $y^2 + 1 = e^{t+\ln 2} = e^t e^{\ln 2} = 2e^t$

So position as a function of time: $y = \sqrt{2e^t - 1}$

(Take positive square root to agree with initial condition $y = 1$ when $t = 0$.)

(iii) Since $\frac{dy}{dt} = \frac{y^2+1}{2y}$, velocity as a function of time:

$$\frac{dy}{dt} = \frac{(2e^t-1)+1}{2(2e^t-1)^{\frac{1}{2}}} = e^t(2e^t-1)^{-\frac{1}{2}}$$