**Question**

Show that the equation $x^2 u_{xx} = y^2 u_{yy}$ is hyperbolic in the positive quadrant and find the equations of the characteristics there, expressing them in the form $f(x, y) = const$, $g(x, y) = const$.

If $u,\ u_x,\ u_y$ are given on the quarter circle $x^2 + y^2 = 1,\ x > 0,\ y > 0$, sketch the domain of the dependence, i.e., the region in the $(x, y)$-plane for which a unique solution of the equation may be determined.

By changing the variables $\xi = f(x, y),\ \eta = g(x, y)$ where $f$ and $g$ are defined above, show that the equation transforms to $2\eta u_{\xi\eta} = u_\xi$. Verify that this has the general solution $u = \alpha(\xi)\sqrt{\eta} + \beta(\eta)$ for arbitrary $\alpha,\ \beta$. Hence deduce the general solution of the original equation in the region $x > 0,\ y > 0$.

**Answer**

$a = x^2,\ b = 0,\ c = -y^2$ so

$b^2 - ac = x^2y^2$ which is $> 0$ everywhere except when $x = 0$ and/or $y = c$

so hyperbolic in positive quadrant.

Characteristics:

$$\frac{dy}{dx} = \pm\frac{xy}{x^2} = \pm\frac{y}{x}$$

$\Rightarrow y = \alpha x, \qquad y = \dfrac{\beta}{x}$

straight lines, hyperbolae

i.e., $\dfrac{y}{x} = const,\ xy = const$

Let $f = \dfrac{y}{x},\ g = xy$ as per question

Boundary conditions are given on $x^2 + y^2 = 1$

From lectures we can calculate Taylor series everywhere within a domain bounded by the characteristic.

PICTURE

From Q7 we suspect $A$ will be important. $A$ is point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Why is it important? The characteristics propagate the boundary condition information from the circle

The domain of dependence can only be defined where there are two characteristics which have come from the boundary condition defining curve, bringing information. If we assume that the boundary conditions are suitably defined on the arc then we have the domain of dependence shown in the following diagram:

PICTURE

Domain of dependence: $g < 1 \;\; x, y > 0$

There is a possible discontinuity along $y = x$.

Since hyperbolic, changing to $\xi = \frac{y}{x}$, $\eta = xy$ we expect to reduce to:

$u_x = -\frac{y}{x^2}u_\xi + yu_\eta$

$u_y = -\frac{1}{x}u_\xi + xu_\eta$

$\xi_x = -\frac{Y}{x}, \;\; \xi_y = \frac{1}{x}, \;\; \eta_x = y, \;\; \eta_y = x$

$$\begin{aligned}
u_{xx} &= \frac{+2y}{x^3}u_\xi + y(u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) - \frac{y}{x}(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x) \\
&= \frac{2y}{x^3}u_\xi - \frac{y^2}{x^2}u_{\eta\xi} + y^2u_{\eta\eta} + \frac{y^2}{x^4}u_{\xi\xi} - \frac{y^2}{x^2}u_{\xi\eta} \\
&= \frac{y^2}{x^4}u_{\xi\xi} - 2\frac{y^2}{x^2}u_{\xi\eta} + y^2u_{\eta\eta} + \frac{2y}{x^3}u_\xi
\end{aligned}$$

$$\begin{aligned}
u_{yy} &= \frac{1}{x}(u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y) + x(u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y) \\
&= \frac{1}{x^2}u_{\xi\xi} + 2u_{\xi\eta} + x^2u_{\eta\eta}
\end{aligned}$$

Thus equation becomes:

$$\frac{y^2}{x^2}u_{\xi\xi} - 2y^2u - \xi\eta + x^2y^2u_{\eta\eta} + 2\frac{y}{x}u_\xi$$

$$= \frac{y^2}{x^2}u_{\xi\xi} + 2y^2u_{\xi\eta} + x^2y^2u_{\eta\eta}$$

i.e., $2\frac{y}{x}u_\xi = 4y^2u_{\xi\eta}$

or $2\xi u_\xi = 4\xi\eta u_{\xi\eta}$

or $2\eta u_{\xi\eta} = u_\xi$ as required.

Substitute in suggested solution $u = \alpha(\xi)\sqrt{\eta} + \beta(\eta)$ to confirm it satisfies equation.

Hence original equation has general solution

$$\underline{u(x,y) = \alpha\left(\frac{y}{x}\right)\sqrt{xy} + \beta(xy)}$$