Question

State which of the following PDEs are elliptic, which are hyperbolic and which are parabolic. Try to find the equations of the characteristics where these are real.

(a) $u_{xx} + 3u_{xy} + u_{yy} = 2u + x^2y^2$

(b) $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$

(c) $x^2u_{xx} - y^2u_{yy} = xy$

(d) $r^2u_{rr} + u_{\theta\theta}$

(e) $\bigtriangledown \cdot [F(r) \bigtriangledown \phi]$

(f) $x(\pi - x)u_{xx} + xu_x + u_t = 0$

Answer

(a) This is an inhomogeneous type with d of th electure notes given as $d = 2u + x^2 - y^2$.

However $a,\ b,\ c$ are given by

$a = 1,\ b = \dfrac{3}{2},\ c = 1 \Rightarrow$ hyperbolic everywhere

Characteristics are given by constant coefficient results,

$$\begin{cases} \xi = y + \left[\dfrac{-3+\sqrt{5}}{2}\right] x \\ \eta = y + \left(\dfrac{-3-\sqrt{5}}{2}\right) x \end{cases}$$

or

$$\begin{cases} y = \left(\dfrac{3-\sqrt{5}}{2}\right) x + const \\ y = \left(\dfrac{3+\sqrt{5}}{2}\right) x + const \end{cases}$$

Note we can solve the homogeneous equation simply as

$$u_{homogeneous}(x,y) = p(\xi) + q(\eta)$$

with $\xi,\ \eta$ above, but complete solution is:

$$u = u_{homogeneous} + u_{particular\ integral}$$

(b) $a = y,\ b = \dfrac{(x+y)}{2},\ c = x$

$b^2 - ac = \dfrac{(x+y)^2}{4} - yx = \left(\dfrac{x-y}{2}\right)^2 \geq 0$

So hyperbolic except where $x = y$ where it's parabolic.

For $y \neq x$ characteristics are given by:

$$a dx^2 + 2b\, dx\, dy + c dy^2 = 0$$

$\Rightarrow y\, dx^2 + (x+y)\, dx\, dy + x dy^2 = 0$

$\Rightarrow$

$$\begin{aligned}\frac{dy}{dx} &= \frac{b \pm \sqrt{b^2 - ac}}{a} \\ &= \frac{(x+y)}{2y} \pm \frac{\sqrt{(x-y)^2}}{2y} \\ &= \frac{1}{2y}[x + y \pm (x-y)] \\ &= \frac{x}{y} \text{ or } \underline{1}\end{aligned}$$

Therefore $\dfrac{dy}{dx} = 1$ or $\dfrac{dy}{dx} = \dfrac{x}{y}$

$\Rightarrow y = x + const$ or $y^2 - x^2 = const$

PICTURE

Note that the two families of characteristics become parallel on $\underline{y = x}$ where equation is parabolic.

(c) $a = x^2,\ b = 0,\ c = -y^2$, inhomogeneous

$b^2 - ac = 0 + x^2y^2 > 0$ for all $x \neq 0,\ y \neq 0$

Characteristics are:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm\frac{xy}{x^2} = \pm\frac{y}{x}$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x},\ \dfrac{dy}{dx} = -\dfrac{y}{x}$

$\Rightarrow \ln y = \ln kx,\ \ln y = -\ln(x\bar{k})$

$\Rightarrow y = kx,\quad \dfrac{\bar{\bar{k}}}{x}$ where k and $\bar{\bar{k}}$ are constants

PICTURE

parallel when $y = 0$ or $x = 0$ i.e., when parabolic

(d) $r^2u_{rr} + u_{\theta\theta} =$ "a"u_r

The "d" is irrelevant to classification

$a = r^2,\ b = 0,\ c = 1$

$b^2 - ac = -r^2 \leq 0 \rightarrow$ elliptic $r \neq 0$, parabolic $r = 0$

No real characteristics for elliptic case

Parabolic case: $\dfrac{dr}{d\theta} = 0 \Rightarrow \underline{r = const}$

(e)

$$\begin{aligned}\nabla \cdot [F(\mathbf{r})\nabla\phi] &= f(\mathbf{r})\nabla^2\phi \\ &= F(\mathbf{r})(\phi_{xx} + \phi_{yy})\end{aligned}$$

so if $F(\mathbf{r})(\phi_{xx}+\phi_{yy}) = 0$ it's elliptic everywhere except where $F(\mathbf{r}) = 0$

$(a = F(\mathbf{r}),\ b = 0,\ c = F(\mathbf{r}),\ b^2 - ac = -F(\mathbf{r})^2 < 0)$

(f) $a = x(\pi - x),\ b = 0,\ c = 0$

(only second order derivatives count)

so $b^2 - ac = 0$ for all $x,\ t$

Thus parabolic with $\dfrac{dt}{dx} = 0 \Rightarrow t = const$