Question

Classify each of the following differential equations below as either hyperbolic, parabolic or elliptic. Find the characteristics, transform the equation to characteristic coordinates and express the general solution in the original variables.

(a) $u_{xx} + 3u_{xy} + 2u_{yy} = 0$

(b) $u_{xx} + 2u_{xy} + 2u_{yy} = 0$

(c) $u_{xx} + 2u_{xy} + u_{yy} = 0$

(d) $u_{xx} + 4u_{xy} - 5u_{yy} = 0$

(e) $u_{xx} + u_{yy} = 0$

(f) $u_{xx} - u_{yy} = 0$

Answer

All are second order linear homogeneous with constant coefficients.

(a) $a = 1,\ b = \frac{3}{2},\ c = 2;\ b^2 - ac = \frac{1}{4} > 0$, hyperbolic.

Characteristics

$$\begin{cases} \xi = y + \left(-\frac{3}{2} + \sqrt{\frac{9}{4} - 2} \right) x \\ \eta = y + \left(-\frac{3}{2} - \sqrt{\frac{9}{4} - 2} \right) x \end{cases}$$

i.e.,

$$\begin{cases} \xi = y - x \\ \eta = y - 2x \end{cases}$$

i.e.,

$$\begin{cases} y = x + const \\ y = 2x + const \end{cases}$$

Transforms to $\frac{\partial^2 u}{\partial \xi}\partial\eta = 0$ in characteristic coordinates

whence $u = f(\xi) + g(\eta)$ in general solution

so $\underline{u(x, y) = f(y - x) + g(y - 2x)}$

CHECK:

$u_x = -f' - 2g' \quad u_{xx} = f'' + 4g''$

$u_y = f' + g' \qquad u_{yy} = f'' + g''$

$\qquad u_{xy} = -f'' - 2g''$

where f' denotes $\dfrac{df(\xi)}{d\xi}$ and g' denotes $\dfrac{dg(\eta)}{d\eta}$ etc.

so

$$\begin{aligned} u_{xx} + 3u_{xy} + 2u_{yy} &= +f'' + 4g'' + 3(-f'' - 2g'') + 2f'' + 2g'' \\ &= 3f'' + 6g'' - 3f'' - 6g'' \\ &= \underline{0}\surd \end{aligned}$$

(b) $a = 1,\ b = 1,\ c = 2;\ b^2 - ac = -1 < 0$ elliptic

No real characteristics. However try the solution

$$\begin{cases} \xi = ay + \left(-b + \sqrt{b^2 - ac}\right) x \\ \eta = ay + \left(-b + \sqrt{b^2 - ac}\right) x \end{cases}$$

i.e.,

$$\begin{cases} \xi = y + (i - 1)x \\ \eta = y + (-i - 1)x \end{cases}$$

NB complex conjugate coefficients of x

Although these are complex characteristics (which we normally reject) we can use these formally as in the hyperbolic case to eliminate the $\dfrac{\partial^2 u}{\partial \xi^2}$ and $\dfrac{\partial^2 u}{\partial \eta^2}$ terms in the transformed (2.1), equation (2.4) of the lecture notes. Instead of obtaining Laplace's equation as in the lecture notes, we obtain $\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$ with solutions $u = p(\xi) + q(\eta)$, i.e.,

$$\underline{u(x, y) = p(y - (1 - i)x) + q(y - (1 + i)x)}$$

This formula could only become real with appropriate boundary conditions (see later). However, we can check that it does satisfy the original equation:

CHECK:

$u_x = p'(i-1) + q'(-i-1) \quad u_{xx} = p''(i-1)^2 + q''(-i-1)^2$

$u_y = p' + q' \qquad u_{yy} = p'' + q''$

$\qquad u_{xy} = p''(i-1) + q''(-i-1)$

$$\begin{aligned} & u_{xx} + 2u_{xy} + 2u_{yy} \\ = \;& p''[(i-1)^2 + 2(i-1) + 2] + q''[(-i-1)^2 + 2(-i-1) + 2] \\ = \;& p''[-1 - 2i + 1 + 2i - 2 + 2] + q''[-1 + x + 1 - 2i - 2 + 2] \\ = \;& \underline{0}\checkmark \end{aligned}$$

NB

Say we carried out the method of the lecture notes and transformed as

$$\begin{cases} \bar{\xi} & = \dfrac{cx - by}{\sqrt{ac - b^2}} & = 2x - y \\ \bar{\eta} & = y & = y \end{cases}$$

The we would obtain Laplace's equation.

$$u_{\bar{\xi}\bar{\xi}} + u_{\bar{\eta}\bar{\eta}} = 0$$

What is the general solution of this? Try the solution

$u = f(\bar{\xi} + i\bar{\eta}) + g(\bar{\xi} - i\bar{\eta})$ $f, \; g$ arbitrary

It <u>works</u>, so the general solution must be

$$\begin{aligned} u(x, y) &= f(2x - y + iy) + g(2x - y - i - y) \\ &= f(2x + (i-1)y) + g(2x + (-i-1)y) \end{aligned}$$

How does this relate to the solution above?

Well $f, \; g, \; p, \; q$ are arbitrary functions.

We can scale the arguments:

$$\begin{aligned} 2x + (i-1)y &= (i-1)\left[\frac{2x}{(i-1)} + y\right] \\ &= (i-1)\left[\frac{2x}{2}(-1-i) + y\right] \\ &= (i-1)[y - (1+i)x] \end{aligned}$$

Similarly

$$\begin{aligned} 2x-(1+i)y &= -(1+i)\left[y-\frac{2x}{(1+i)}\right] \\ &= -(1+i)[y-(1-i)x] \end{aligned}$$

So if we let

$$\begin{cases} f((i-1)\bar{X}) = p(\bar{X}) \\ g(-(1+i)\bar{Y}) = p(\bar{X}) \end{cases}$$

we obtain

$$\underline{u(x,y) = p(y-(1-i)x) + q(y-(1+i)x)}$$

In $x,\ y$ coords, the same general solution as <u>above</u> $\surd$.

MORAL:

General solution to Laplace equation

$u_{\bar{\xi}\bar{\xi}} + u_{\bar{\eta}\bar{\eta}} = 0$ is $u = f(\bar{\xi}+i\bar{\eta}) + g(\bar{\xi}-i\bar{\eta})$

(c) $a=1,\ b=1,\ c=1;\ b^2-ac=0$, parabolic.

Characteristics: only one set

$$\begin{cases} \xi = ay - bx \leftarrow \text{ characteristic} \\ \eta = y \leftarrow \text{ not important} \end{cases}$$

i.e.,

$$\begin{cases} \xi = y - x \\ \eta = y \end{cases}$$

Transforms equation to $\dfrac{\partial^2 u}{\partial \eta^2} = 0$

whence $u(\xi,\eta) = p(\xi) + \eta q(\xi)$ is general solution

$$\Rightarrow \underline{u(x,y) = p(y-x) + yq(y-x)}$$

(d) $a = 1,\ b = 2,\ c = -5;\ b^2 - ac = 9 > 0$, hyperbolic.

Characteristics

$$\begin{cases} \xi = y + x \\ \eta = y - 5x \end{cases}$$

i.e.,

$$\begin{cases} y = -x + const \\ y = 5x + const \end{cases}$$

Gives general solution to transformed equation $u_{\xi\eta} = 0$ as $u = p(\xi) + q(\eta)$, so

$$\underline{u(x, y) = p(y + x) + q(y - 5x)}$$

(e) $u_{xx} + u_{yy} = 0,\ a = 1,\ b = 0,\ c = 1;\ b^2 - ac = -1 < 0$ $\underline{\text{elliptic}}$

For general solution see (B) above.

For solution with boundary conditions given see Q7

(f) $u_{xx} - u_{yy} = 0,\ a = 1,\ b = 0,\ c = -1;\ b^2 - ac = 1 > 0$ $\underline{\text{hyperbolic}}$

Characteristics

$$\begin{cases} \xi = y + x \\ \eta = y - x \end{cases}$$

So general solution is

$$\underline{u(x, y) = p(y + x) + q(y - x)}$$