Question
Prove the standard forms for elliptic, parabolic and hyperbolic second order linear PDEs with constant coefficients of type

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0$$

as derived in lectures. Hence check the form of their general solutions given in lectures.

Answer
Use lecture notes with the transformations

$$\begin{cases} \xi = \alpha y + \beta x \\ \eta = \gamma y + \delta x \end{cases}$$

where $\alpha,\ \beta,\ \gamma,\ \delta = const$
These give rise to
"$\partial_x = \dfrac{\partial}{\partial x}$" $\rightarrow \partial_x = \dfrac{\partial \xi}{\partial x}\partial_\xi + \dfrac{\partial \eta}{\partial x}\partial_\eta = \beta\partial_\xi + \delta\partial_\eta$
"$\partial_y = \dfrac{\partial}{\partial y}$" (shorthand) $\rightarrow \partial_{\mathrm{y}} = \dfrac{\partial \xi}{\partial \mathrm{y}}\partial_\xi + \dfrac{\partial \eta}{\partial \mathrm{y}}\partial_\eta = \alpha\partial_\xi + \gamma\partial_\eta$
$\Rightarrow \partial_x^2 = (\beta\partial_\xi + \delta\partial_\eta)^2 = \beta^2\partial_\xi^2 + 2\beta\delta\partial_{\xi\eta}^2 + \delta^2\partial_\eta^2$
Note: Would have to be more careful if β and δ were <u>not</u> constants
$\partial_{xy}^2 = (\beta\partial_\xi + \delta\partial_\eta)(\alpha\partial_\xi + \gamma\partial_\eta) = \alpha\beta\partial_\xi^2 + (\alpha\delta + \gamma\beta)\partial_{\xi\eta}^2$
$\Rightarrow \partial_y^2 = (\alpha\partial_\xi + \gamma\partial_\eta)^2 = \alpha^2\partial_\xi^2 + 2\alpha\gamma\partial_{\xi\eta}^2 + \gamma^2\partial_\eta^2$
$\Rightarrow au_{xx} + 2bu_{xy} + cu_{yy} = 0$ becomes

$$\begin{aligned}(a\beta^2 + 2b\alpha\beta + c\alpha^2) \quad &+ \quad (2a\beta\delta + 2b(\alpha\delta + \gamma\beta) + 2c\alpha\delta)u_{\xi\eta} \\ &+ \quad (a\partial^2 + 2b\gamma\delta + c\gamma^2)u_{\eta\eta} = 0\end{aligned}$$

3 cases:

$$\left.\begin{array}{rrcl} (i) & \text{hyperbolic} & : & b^2 > ac \\ (ii) & \text{parabolic} & : & b^2 = ac \\ (iii) & \text{elliptic} & : & b^2 < ac \end{array}\right\} \text{see lecture notes.}$$

These give $(\star)$ as:

(i) $\dfrac{\partial^2 u}{\partial\xi\partial\eta} = 0 \Rightarrow \text{(integrate)}\dfrac{\partial \mathrm{u}}{\partial\xi} = \mathrm{p}'(\xi) \Rightarrow \text{(integrate)}\mathrm{u} = \mathrm{p}(\xi) + \mathrm{q}(\eta)$

(ii) $\dfrac{\partial^2 u}{\partial\eta^2} = 0 \Rightarrow \text{(integrate)}\dfrac{\partial \mathrm{u}}{\partial\eta} = \mathrm{q}(\xi) \Rightarrow \text{(integrate)}\mathrm{u} = \mathrm{p}(\xi) + \eta\mathrm{q}(\xi)$

(iii) $\dfrac{\partial^2 u}{\partial\eta^2} + \dfrac{\partial^2 u}{\partial\eta^2} = 0 \Rightarrow$ solutions outlines later in the problem sheet

Remember can treat $p(\xi)$, $q(\eta)$ etc. as arbitrary functions of ξ, η (cf. constants in ODEs)