Exam Question
Topic: DiffInt
Show that $f(x)$ is an odd function, where

$$f(x) = \int_x^{2x} \exp\left(-t^2\right)\, dt.$$

By differentiating the integral find the turning points of $f(x)$ and identify their type.

Solution
Writing down the formula for $f(-x)$ and substituting $t = -u$ gives

$$f(-x) = \int_{-x}^{-2x} \exp(-t^2)\, dt = \int_x^{2x} \exp(-u^2)\,(-du) = -f(x).$$

$$\begin{aligned} f'(x) &= 2\exp(-4x^2) - \exp(-x^2) \\ &= 2\exp(-x^2)\left(\exp(-3x^2 - \frac{1}{2}\right) \\ &= 0 \text{ iff } \exp(-3x^2) = \frac{1}{2} \text{ i.e. } x^2 = \frac{\ln 2}{3}. \end{aligned}$$

$$\begin{aligned} \text{For } x > 0,\ f'(x) &> 0 \text{ if } x < \sqrt{\frac{\ln 2}{3}}. \\ f'(x) &< 0 \text{ if } x > \sqrt{\frac{\ln 2}{3}} \end{aligned}$$

So f has a maximum at $x = \sqrt{\frac{\ln 2}{3}}$ and hence, since f is odd, a minimum at $x = -\sqrt{\frac{\ln 2}{3}}$