Exam Question
Topic: DiffInt
The function $f$ satisfies the equation

$$f(x) = \frac{\pi}{3} + \int_0^x \cos^2(f(t)) \, dt.$$ 

By differentiating the integral, find a differential equation for $f(x)$. Hence find the function $f(x)$.

Solution

$$f(x) = \frac{\pi}{3} + \int_0^x \cos^2(f(t)) \, dt.$$ 

$$\frac{df}{dx} = \cos^2(f(x)), \text{ so } \sec^2(f) \frac{df}{dx} = 1.$$ 

Therefore $\tan(f(x)) = x + c$.

When $x = 0$, $f(x) = \frac{\pi}{3}$, so $c = \sqrt{3}$.

Thus $f(x) = \tan^{-1}\left(x + \sqrt{3}\right)$. 

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