Exam Question
Topic: DiffInt
The function $f$ satisfies the equation

$$f(x) = \sqrt{3} + \int_0^x [1 + (f(t))^2] \, dt.$$ 

By differentiating the integral, find a differential equation for $f(x)$. Hence find the function $f(x)$.

Solution

$$f(x) \sqrt{3} + \int_0^x [1 + (f(t))^2] \, dt$$

Differentiating with respect to $x$ gives

$$\frac{df}{dx} = 1 + (f(x))^2 \quad \text{so} \quad \frac{1}{1 + f^2} \frac{df}{dx} = 1.$$ 

This gives $\tan^{-1} f = x + c$, so $f(x) = \tan(x + c)$. 
Now $f(0) = \tan c = \sqrt{3}$ so $c = \pi/3$. Hence $f(x) = \tan \left( x + \frac{\pi}{3} \right)$.