QUESTION

Consider a portfolio of 4 risky assets $X^{(1)},\ X^{(2)}$ which are held in proportion θ_1 and $\theta_2 = 1 - \theta_1$. Let them be distributed normally with $\mu_1,\ \mu_2$ and variance $\sigma_1^2,\ \sigma_2^2$ respectively.

(a) Show that the mean value of the portfolio is $\mu = \theta_1\mu_1 + \theta_2\mu_2$.

(b) Show that if the prices of the risky assets are uncorrelated, then the variance of the portfolio is given by $\theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2$.

(c) Show that if the prices of the risky assets have some correlation then the variance is given by $\theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2 + 2\theta_1\theta_2\rho_{12}$ where the correlation between the prices $\rho_{12} = \langle X(1)X(2)\rangle - \mu_1\mu_2$.

(d) Given the following data evaluate the mean and variance of the portfolio: $\mu_1 - 0.2,\ \sigma_1 = 0.75,\ \mu_2 = 0.16,\ \sigma_2 = 0.5,\ \rho_{12} = -0.60$.

ANSWER

$$\begin{array}{ll} X^{(1)} : \theta_1 & X^{(1)} \in N(\mu_1, \sigma_1^2) \\ X^{(2)} : \theta_2 = 1 - \theta_1 & X^{(2)} \in N(\mu_1, \sigma_2^2) \end{array}$$

(a) $\left\langle \theta_1 X^{(1)} + \theta_2 X^{(2)} \right\rangle$ is mean value of portfolio$= \mu = \theta_1 \left\langle X^{(1)} \right\rangle + \theta_2 \left\langle X^{(2)} \right\rangle = \theta_1\mu_1 + \theta_2\mu_2 = \mu$

(b) Variance

$$\begin{aligned}
\sigma^2 &= \left\langle [(\theta_1 X^{(1)} + \theta_2 X^{(2)}) - \mu]^2 \right\rangle \\
&= \left\langle (\theta_1(X^{(1)} - \mu_1) + \theta_2(X^{(2)} - \mu_2))^2 \right\rangle \\
&= \left\langle \theta_1^2 \underbrace{(X^{(1)} - \mu_1)^2}_{<>=\sigma_1^2} + \theta_2^2 \underbrace{(X^{(2)} - \mu_2^2)}_{<>=\sigma_2^2} + 2\theta_1\theta_2 (X^{(1)} - \mu_1) \times (X^{(2)} - \mu_2) \right\rangle \\
&= \theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2 + 2\theta_1\theta_2 \underbrace{\left\langle (X^{(1)} - \mu_1)(X^{(2)} - \mu_2) \right\rangle}_{\text{correlation term}=0} \\
&= \theta_1^2\sigma_1^2 + \sigma_2^2\theta_2^2
\end{aligned}$$

(c) If correlation $\neq 0$ must include

$$\begin{aligned}
&= 2\theta_1\theta_2 \left\langle (X^{(1)} - \mu_1)(X^{(2)} - \mu_2) \right\rangle \text{ term} \\
&= 2\theta_1\theta_2 \left\langle x^{(1)} X^{(2)} - \mu_1 X^{(2)} - \mu_2 X^{(1)} + \mu_1\mu_2 \right\rangle
\end{aligned}$$

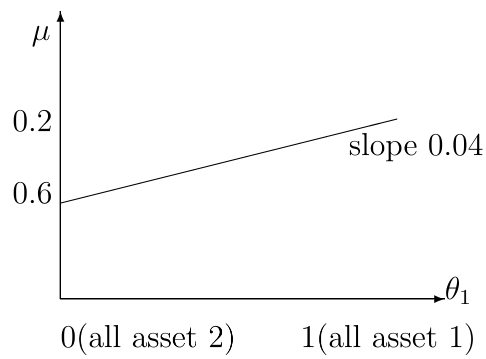
$$
\begin{aligned}
&= 2\theta_1\theta_2\Big[\Big\langle X^{(1)}X^{(2)}\Big\rangle - \mu_1\underbrace{\Big\langle X^{(2)}\Big\rangle}_{=\mu_2} - \mu_2\underbrace{\Big\langle X^{(1)}\Big\rangle}_{=\mu_1} + \mu_1\mu_2\Big] \\
&= 2\theta_1\theta_2\underbrace{\Big[\Big\langle X^{(1)}X^{(2)}\Big\rangle - \mu_1\mu_2\Big]}_{=\rho_{12}}
\end{aligned}
$$

Hence

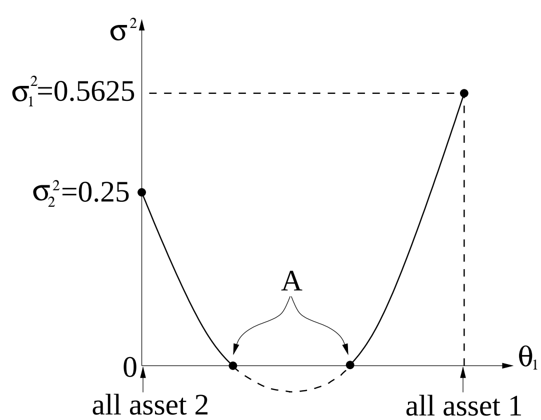
$$\sigma^2 = \theta_1^2\sigma_1^2 + \sigma_2^2\sigma_2^2 + 2\rho_{12}\theta_1\theta_2$$

(d) $\left.\begin{array}{ll}\mu_1 = 0.2, & \sigma_1 = 0.75 \\ \mu_2 = 0.16, & \sigma_2 = 0.5\end{array}\right\}\rho_{12} = 0.60.$

$\mu = 0.2\theta_1 + 0.16\theta_2 = 0.2\theta_1 + 0.16(1-\theta_1) = 0.16 + 0.04\theta_1$



$$
\begin{aligned}
\sigma^2 &= \theta_1(0.75)^2 + (0.5)^2\theta_2^2 - 2\times 0.6\times\theta_1\theta_2 \\
&= \theta_1^2(0.5625) + 0.25(1-\theta_2)^2 - 1.2\theta_1(1-\theta_1) \\
&= 2.0125\theta_1^2 - 1,70\theta_1 + 0.25
\end{aligned}
$$



Use combination θ_1, θ_2 from region A to eliminate σ^2, i.e. minimise variance and osciallations in portfolio values.

$\sigma^2 = 0$ when $\theta_1 = \frac{1.70+\sqrt{1.7^2-4\times 2.0125\times 0.25}}{2\times 2.0125} = 0.6551$ or 0.1896.

Better to pick the 0.6551 value since then μ is higher

$\mu = 0.16 + 0.04 \times 0.6551 - 0.1862$