Question

(a) Find the constants a,b,c so that the force feild defined by

$$(x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$$

is conservative.

(b) What is the potential associated with this force field when a, b, c are chosen to make it conservative.

Answer
We require $\nabla \times \mathbf{F} = 0$

$$\begin{aligned} 0 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} \\ &= \mathbf{i}(c+1) + \mathbf{j}(a-4) + \mathbf{k}(b-2) \end{aligned}$$

Therefore $a = 4, b = 2, c = -1$.

$$\begin{aligned} \mathbf{F} &= (x + 2y + 4z)\mathbf{i} + (2x - 3y - z)\mathbf{j} + (4x - 1y + 2z)\mathbf{k} \\ &= -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j} - \frac{\partial U}{\partial z}\mathbf{k} \end{aligned}$$

$$\begin{aligned} -\frac{\partial u}{\partial x} &= -(x + 2y + 4z) \\ \Rightarrow U &= -\tfrac{1}{2}x^2 - 2xy - 4xz + f_1(y, z) \quad (1) \end{aligned}$$

$$\begin{aligned} -\frac{\partial U}{\partial y} &= -(2x - 3y - 2) \\ \Rightarrow U &= -2xy + \tfrac{3}{2}y^2 + yz + f_2(x, z) \quad (2) \end{aligned}$$

$$\begin{aligned} -\frac{\partial U}{\partial z} &= -(4x - y + 2z) \\ \Rightarrow U &= -4xz + yz + \tfrac{1}{2}z^2 + f_3(x, z) \quad (3) \end{aligned}$$

Comparing:

$$\begin{aligned} (1) \text{ and } (2) \ \Rightarrow \ f_1(y, z) &= \tfrac{3}{2}y^2 + yz + g(z) \\ f_2(x, z) &= -4xz - \tfrac{1}{2}x^2 g(z) \\ (2) \text{ and } (3) \ \Rightarrow \ f_3(x, y) &= -2xy + \tfrac{3}{2}y^2 - \tfrac{1}{2}x^2, g(z) = -\tfrac{1}{2}z^2 \end{aligned}$$

Therefore $U = -4x^2 + y^2 - 2xy - \frac{1}{2}x^2 + \frac{3}{2}y^2 - \frac{1}{2}z^2$