Question

(a) Prove that the force field

$$\mathbf{F} = (y^2 - 2xyz^3)\mathbf{i} + (3 + 2xy - x^2z^3)\mathbf{j} + (6z^3 - 3x^2yz^2)\mathbf{k}$$

is conservative.

(b) Find the potential $U(x, y, z)$ associated with the force field.

(c) Find the work done by the field in moving a particle from the point (2,-1,2) to (-1,3,-2)

Answer

(a)

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - 2xyz^3 & 3 + 2xy - x^2z^3 & 6z^3 - 3x^2yz^2 \end{vmatrix} \\
&= \mathbf{i}(-3x^2z^2 - 3x^2z^2) - \mathbf{j}(6xyz^2 - 6xyz^2) \\
&\quad + \mathbf{k}(2y - 2x^2z^3 - 2y + 2xz^3) \\
&= 0
\end{aligned}$$

Therefore $\mathbf{F}$ is conservative.

(b)

$$\mathbf{F} = -\nabla U = -\mathbf{i}\frac{\partial U}{\partial x} - \mathbf{j}\frac{\partial U}{\partial y} - \mathbf{k}\frac{\partial U}{\partial z}$$

Equating components:

$$\begin{aligned}
-\frac{\partial U}{\partial x} &= y^2x - 2xyz^3 \\
\Rightarrow U &= -y^2 + x^2yz^3 + f_1(y, z) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial U}{\partial y} &= 3 + 2xy - x^2z^3 \\
\Rightarrow U &= -3y + x^2y^2 - yx^2z^2 + f_2(x, z) \quad (2)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial U}{\partial z} &= 6z^3 - 3x^2yz^2 \\
\Rightarrow U &= -\tfrac{3}{2}z^4 + x^2yz^3 + f_3(x, z) \qquad (3)
\end{aligned}$$

Comparing:

$$\begin{array}{lcrcll} (1) \text{ and } (2) & \Rightarrow & f_1(y,z) & = & -3y+g(z), & f_2 = g(z) \\ (2) \text{ and } (3) & \Rightarrow & g(z) & = & -\frac{3}{2}z^4, & f_3 = -3y - xy^2 \end{array}$$

Therefore $U = -y^2x + x^2yz^3 - 3y - \frac{3}{2}z^4(+\text{constant})$

(c) Work done is the difference in $-U$

$$\begin{aligned} U(2,-1,2) &= (-1)^2 2 + 2^2(-1)2^2 - 3(-1)\frac{3}{2}2^4 \\ &= -2 - 32 + 3 - 42 \\ &= -55 \\ U(-1,3,-2) &= (-9)(-1) + (-1)^2 3(-2)^3 - 33\frac{3}{2}(-2)^4 \\ &= -9 - 24 - 9 - 24 \\ &= -48 \end{aligned}$$

Work done is $-(-48 - -55) = -7$