Question

Let X and Y denote the scores of two class tests for a randomly selected student, called Miss T. Assume that X and Y is bivariate normal $N_2(\mu_x = 85, \mu_y = 90, \sigma_x = 10, \sigma_y = 16, \rho = 0.8)$.

(a) What is the probability that the sum of her score on the first two tests will be greater than 200?

(b) What is the probability that her score on the first test (X) will be higher than her score on the second test?

(c) If Miss T's score X on the first test is 80, what is the probability that her score on the second test will be higher than 90?

Answer

(a) Let $W = X + Y$

$$\begin{aligned} E(W) &= E(X) + E(Y) \\ &= \mu_x + \mu_y = 85 + 90 = 175 \end{aligned}$$

$$\begin{aligned} \text{var}(W) &= \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y) \\ &= 10^2 + 16^2 + 2(0.8)(10)(16) \\ &= 612 \end{aligned}$$

Since W is a linear combination of X and Y

$$W \sim N(175, 612)$$

Therefore

$$\begin{aligned} P(W > 200) &= P\left\{\frac{W - 175}{\sqrt{612}} > \frac{200 - 175}{\sqrt{612}}\right\} \\ &= 1 - \Phi\left(\frac{25}{\sqrt{612}}\right) = 0.1562 \end{aligned}$$

(b) Let $W = X - Y$

Therefore $E(W) = 85 - 90 = -5$

$\text{var}(W) = 10^2 + 16^2 - 2(0.8)(10)(16) = 100$

$P(W > 0) = P\left\{\frac{W-(-5)}{10} > \frac{0-(-5)}{10}\right\} = 1 - \Phi(\frac{1}{2}) = 0.3085$

(c) $P(Y > 90|X = 80)$

We need the distribution of $Y|X = x$.

$E(Y|X = x) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$

$\text{var}(Y|X = x) = \sigma_y^2(1 - \rho^2) = 16^2(1 - 0.8^2) = 92.16$

Therefore $E(Y|X = x) = 90 + 0.8\frac{16}{10}(80 - 85) = 83.6$

Therefore

$$\begin{aligned} P(Y > 90|X = 80) &= P\left\{\frac{Y - 83.6}{9.6} > \frac{90 - 83.6}{9.6}\middle| X = 80\right\} \\ &= P\left(Z > \frac{6.4}{9.6}\right) \text{ where } Z \sim N(0,1) \\ &= 0.2525 \end{aligned}$$