Question

a) Find the images, in the w-plane, of lines parallel to the real and imaginary axes in the z-plane, under the transformation $w = e^z$. Explain how this illustrates the concept of conformality.

b) Show that any Mobius transforamtion mapping the upper half plane $\text{im}(z) \geq 0$ into the upper half plane $\text{im}(w) \geq 0$ must be of the form

$$w = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where α, β, γ, δ are all real and $\alpha\delta - \beta\gamma > 0$. Deduce the general form of Mobius transformation mapping $\text{im}(z) \geq 0$ onto the right hand half plane $\text{re}(w) \geq 0$.

Answer

a) Let $w = e^z$ and write $z = x + iy$. Then $w = e^{x+iy} = e^x e^{iy}$.

For x constant, as y varies over $\mathbf{R}$, w traces round the circle centre O radius e^x in the w plane, infinitely many times.

For y constant, as x varies over $\mathbf{R}$, w traces the ray from O (but not including O) making an angle y $(\text{mod}2\pi)$ with the positive real axis.

Lines parallel to the real and imaginary axes in the z-plane are orthogonal, as are circles centre O and rays from O in the w-plane.

This illustrates the angle-preserving property which is that of conformality.

b) The required transformations must map the real axis to the real axis (boundary $\to$ boundary).

Thus a pair of finite non-real conjugate points map to a pair of finite non-real conjugate points. Hence $z = \infty$ maps onto the real axis and $w = \infty$ is the image point on the real axis.

i) If $\infty \to \infty$ then $C = 0$, so $d \neq 0$ and $w = \alpha z + \beta$.
$z = 0 \to$ real w so β is real.
then $z = 1 \to$ real w so α is real.

ii) If $C \neq 0$, $w = \dfrac{Az+B}{z+D}$

$z = -D \to w = \infty$ so $-D$ must be real.

$D = 0 \Rightarrow w = A + \dfrac{B}{z}$

$z = \infty \to$ real w so A is real.

then $z = 1 \to$ real w so B is real.

$D \neq 0 \Rightarrow z = 0 \to \frac{B}{D}$ - real so B is real.

then $z = 1 \to w$ real so A is real.

Thus in all cases the transformation has the form

$w = \dfrac{\alpha z + \beta}{\gamma z + \delta}, \;\; \alpha, \;\beta, \;\gamma, \;\delta$ real

when $z = i$, im$w > 0$,

$\dfrac{\alpha z + \beta}{\gamma z + \delta} = \dfrac{(\alpha\delta - \beta\gamma)i + (\alpha\gamma + \beta\delta)}{\gamma^2 + \delta^2}$ so $\alpha\delta - \beta\gamma > 0$.

Now $w = e^{-i\frac{\pi}{2}} \dfrac{\alpha z + \beta}{\gamma z + \delta}$ maps im$z \geq 0$ onto re$z \geq 0$.

Conversely if $w = \dfrac{az+b}{cz+d}$ maps im$z \geq 0$ to re$w \geq 0$ then $w = e^{i\frac{\pi}{2}} \dfrac{az+b}{cz+d}$ maps im$z \geq 0$ to im$w \geq 0$.

So $e^{i\frac{\pi}{2}} a = \alpha$- real

$a = \alpha e^{-i\frac{\pi}{2}}$

So $w = e^{-i\frac{\pi}{2}} \dfrac{\alpha z + \beta}{\gamma z + \delta}$

$\alpha, \;\beta, \;\gamma, \;\delta$ are all real and $\alpha\delta - \beta\gamma > 0$.