Question

a) Let $f(z)$ have a pole of order 2 at $z = z_0$, and write $g(z) = (z-z_0)^2 f(z)$. Show that the residue of f at z_0 is $g'(z_0)$.

Evaluate, by contour integration

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx$$

where a is a positive real number.

b) Evaluate, by contour integration

$$\int_0^{2\pi} \frac{d\theta}{8\cos^2\theta + 1}.$$

Answer

a) $f(z)$ has a pole of order 2 at $z = z_0$, so the Laurent expansion in a neighbourhood of z_o gives

$f(z) = \dfrac{\alpha}{(z-z_0)^2} + \dfrac{\beta}{(z-z_0)} + \phi(z)$

where $g(z)$ is analytic in the neighbourhood.

So $(z-z_0)^2 f(z) = \alpha + \beta(z-z_0) + (z-z_0)^2\phi(z)$

$\dfrac{d}{dz}((z-z_0)^2 f(z)) = \beta + 2(z-z_0)g(z) + (z-z_0)^2\phi(z)$

Putting $z = z_0$ gives the result.

Let $f(z) = \dfrac{z^2}{(z^2+a^2)^2}$, this has a pole of order 2 at $z = ia$ in the upper half plane.

Its residue is given by

$\left.\dfrac{d}{dz}(z-ia)^2 f(z)\right|_{z=ia} = \dfrac{d}{dz}\left(\dfrac{z^2}{(z+ia)^2}\right)_{z=ia}$

$$\left.\frac{(z+ia)^2 2z - z^2 2(z+ia)}{(z+ia)^4}\right|_{z=ia} = \frac{4(ia)^3}{16(ia)^4} = \frac{1}{4ia}$$

We integrate $f(z)$ round the contour Γ comprising the line segment $(-R, R)$ on the real axis, and the semicircle $C = \{z = Re^{it} \ \ 0 \le t \le \pi\}$ where $R > a$.

Then on C $\qquad |f(z)| = \dfrac{|z|^2}{|z^2+a^2|^2} \le \dfrac{r^2}{(R^2-a^2)^2}$

and so $\left|\int_C f(z)dz\right| \le \dfrac{R^2}{(R^2-a^2)^2}\pi R \to 0$ as $R \to \infty$.

Now $\int_\Gamma f(z)dz = 2\pi i \dfrac{1}{4ia} = \dfrac{\pi}{2a}$

and $\int_\Gamma f(z)dz = \int_C f(z)dz + \int_{-R}^{R} fxdx$

Letting $R \to \infty$ gives $\int_{-\infty}^{\infty} f(x)dx = \dfrac{\pi}{2a}$

b) Let $z = e^{i\theta}$ and C be the unit circle, then $dz = ie^{i\theta}d\theta$ so $d\theta = \frac{dz}{iz}$

$\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. Hence we have

$$I = \int_0^{2\pi} \frac{d\theta}{8\cos^2\theta + 1} = \int_C \frac{dz}{iz\left(8\frac{1}{4}\left(z+\frac{1}{z}\right)^2 + 1\right)}$$

$$= \frac{1}{i}\int_C \frac{dz}{z\left(2z^2 + 4 + \frac{2}{z^2} + 1\right)} = \frac{1}{i}\int_C \frac{zdz}{2z^4 + 5z^2 + 2}$$

$$= \frac{1}{2i}\int_C \frac{zdz}{\left(z^2 + \frac{1}{z}\right)(z^2+2)}$$

Let $f(z) = \dfrac{z}{\left(z^2 + \frac{1}{z}\right)(z^2+2)}$

Then $f(z)$ has simple poles at $z = \pm\frac{i}{\sqrt{2}}$ inside C.

$$\mathrm{Res}\left(f, \frac{i}{\sqrt{2}}\right) = \lim_{z \to \frac{i}{\sqrt{2}}} \left(z - \frac{i}{\sqrt{2}}\right) f(z) = \lim_{z \to \frac{i}{\sqrt{2}}} \frac{z}{\left(z + \frac{i}{\sqrt{2}}\right)(z^2+2)}$$

$$= \frac{\frac{i}{\sqrt{2}}}{\frac{2i}{\sqrt{2}}\frac{3}{2}} = \frac{1}{3}$$

$$\text{Res}\left(f, \frac{-i}{\sqrt{2}}\right) = \lim_{z \to -\frac{i}{\sqrt{2}}} \left(z + \frac{i}{\sqrt{2}}\right) f(z) = \lim_{z \to -\frac{i}{\sqrt{2}}} \frac{z}{\left(z - \frac{i}{\sqrt{2}}\right)(z^2 + 2)}$$

$$= \frac{-\frac{i}{\sqrt{2}}}{-\frac{2i}{\sqrt{2}}\frac{3}{2}} = \frac{1}{3}$$

So $I = \frac{1}{2i} \int_C f(z) dz = \frac{1}{2i} 2\pi i \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2\pi}{3}$