Question

a) Let C denote the spiral

$$\{z : z = e^{\theta^2 + i\theta} \;\; 0 \leq \theta \leq 4\pi\}.$$

Evaluate $\int_C \frac{dz}{z}$, giving a careful explanation of your reasoning.

b) Use Cauchy's Integral Formula to evaluate

$$\int_C \frac{e^z \sin^2 \pi z}{(z-1)^3} dz$$

where C is a simple closed contour having $z = 1$ in its interior.

Answer

a) On C, $z = e^{\theta^2} e^{i\theta}$

$z = e^{\theta^2}$ increase so $|z|$ increase with θ. So we have a spiral

DIAGRAM

We need to use the logarithmic function, but we have to divide C up into subsets on which $\arg z$ changes by less than 2π,

on C_1 $\quad 0 \leq \arg z \leq \frac{3\pi}{2}$

on C_2 $\quad \frac{3\pi}{2} \leq \arg z \leq 3\pi$

on C_3 $\quad 3\pi \leq \arg z \leq 4\pi$

on C_1 $\quad \frac{d}{dz} \log_{\frac{3\pi}{4}} z = \frac{1}{z}$

on C_2 $\quad \frac{d}{dz} \log_{\frac{9\pi}{4}} z = \frac{1}{z}$

on C_3 $\quad \frac{d}{dz} \log_{\frac{7\pi}{2}} z = \frac{1}{z}$

so $\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} + \int_{C_3} \frac{dz}{z}$

$$= [\log|z| + i\arg z]_{C_1} + [\log|z| + i\arg z]_{C_2} + [\log|z| + i\arg z]_{C_3}$$

Because $\arg z$ has been chosen continuously on C, this reduces to

$[\log|z| + i\arg z]_C = \log e^{(4\pi)^2} - \log e^0 + i4\pi = 16\pi^2 + 4\pi i$

Or this could be done directly from the definition of the integral which is rather easier, as follows

$\int_C \frac{dz}{z} \qquad z = e^{\theta^2 + i\theta} \qquad dz = (2\theta + i)e^{\theta^2 + i\theta}$

so $\int_C \frac{dz}{z} = \int_0^{4\pi} 2\theta + i = [\theta^2 + i\theta]_0^{4\pi} = 16\pi^2 + 4\pi i$

b) $\int_C \frac{e^z \sin^2 \pi z}{(z-1)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}(e^z \sin^2 \pi z)_{z=1}$

$\frac{d}{dz} e^z \sin^2 \pi z = e^z \sin^2 \pi z + e^z 2 \sin \pi z.\pi \cos \pi z$

$= e^z \sin^2 \pi z + e^z \pi \sin 2\pi z$

$\frac{d^2}{dz^2} e^z \sin^2 \pi z = e^z \sin^2 \pi z + e^z \pi \sin 2\pi z + e^z \pi \sin 2\pi z + e^z 4\pi^2 \cos 2\pi z$

At $z = 1$

$\sin \pi z = 0 \qquad \sin 2\pi z = 0 \qquad \cos 2\pi z = 1$

so the integral is $\frac{2\pi i}{2!} 2\pi^2 e = 2\pi^3 e i$