Question

The plane curve $\alpha(t) = (2 + \cos t, \sin t)$ is a circle. The space curve

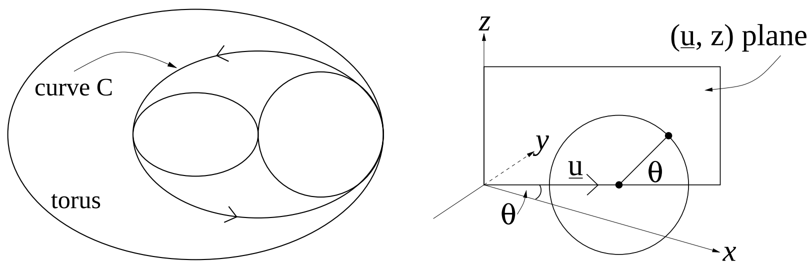
$$\gamma(t) = ((2 + \cos t)\cos t, (2 + \cos t)\sin t, \sin t)$$

lies on a torus, thought of as being swept out by α as the plane of α is spun around the z-axis in R^3. Show that the curvature of γ vanishes at $(-1, 0, 0)$. Find the curvature and the torsion of γ at the point $(3, 0, 0)$, and find the equation of the osculating plane there.

Answer

If $\underline{u} = (a, b)$ is a unit vector in the (x, y)-plane, then $((2 + c)a, (2 + c)b, s)$ is a point on a circle in the $(\underline{u}, x)$-plane (centre $2\underline{u}$, radius 1).

Thus $\gamma(t)$ is on the torus swept out by these circles as $\underline{u}$goes around the unit circle in the (x, y)-plane.



$$\begin{aligned}\gamma'(t) &= (-2\sin t - \sin 2t, 2\cos t + \cos 2t, \cos t)\\ \gamma''(t) &= (-2\cos t - 2\cos 2t, -2\sin t - 2\sin 2t, -\sin t)\end{aligned}$$

$$\begin{aligned}K = 0 \;&\Rightarrow\; x'y'' - x''y' = 0\\ \textbf{i.e. } (2\sin t + \sin 2t)(-2\sin t - 2\sin 2t) &= *2\cos t + 2\cos 2t)(2\cos t + \cos 2t)\\ \Rightarrow -6 &= 6(\cos t\cos 2t + \sin t \sin 2t)\\ \textbf{i.e. } -1 &= \cos t\\ t &= \pi \quad (+2n\pi, n \in \mathbf{Z})\end{aligned}$$

So curvature vanishes at $(-1, 0, 0)$ (as C passes through the inner "equator").

$$\gamma'''(t) = (2\sin t + 4\sin 2t, -2\cos t - 4\cos 2t, -\cos t)$$

and at $(3, 0, 0)$ (i.e $t = 0$) we have

$$\begin{aligned}\gamma' &= (0, 3, 1)\\ \gamma'' &= (-4, 0, 0)\\ \gamma''' &= (0, -6, -1)\\ \gamma' \cap \gamma'' &= (0, -4, 12)\end{aligned}$$

There, $\tau = \dfrac{\gamma' \cap \gamma''.\gamma'''}{\|\gamma' \cap \gamma''\|^2} = \dfrac{12}{160} = \dfrac{3}{40}$
The binomial B **is in the direction of** $\gamma' \cap \gamma''$**, so**

$$B = \frac{1}{\sqrt{160}}(0, -4, 12) = \frac{1}{\sqrt{10}}(0, -1, 3).$$

Osculating plane is $\perp B$ **so has the equation** $0x - y + 3z =$ **constant** $= 0$**, since it contains the point** $(3, 0, 0)$**.**
So OSC plane is: $y = 3z$**.**