Question

Let

$$\begin{aligned}\gamma_1(t) &= (\cosh 2t + 2\cosh t, 2\sinh t - \sinh 2t)\\ \gamma_2(t) &= (\cosh 2t - 2\cosh t, 2\sinh t - \sinh 2t).\end{aligned}$$

Show that γ_2 is a regular curve, while γ_1 has a singularity at the point $(3,0)$. Sketch γ_1 and γ_2. Find the centre of curvature of γ_2 at the point $(-1,0)$.
[*Exploit symmetry, and look at the directions of the tangents to the curves.*]

Answer

Write $\gamma(t) = (\cosh 2t + 2\epsilon\cosh t, 2\sinh t - \epsilon\sinh 2t)$
Where $\epsilon = \pm 1$ **(so** γ_i **has** $\epsilon = (-1)^{i+1}$**)**
So $\dot{\gamma}(t) = (2sinh2t + 2\epsilon\sinh t, 2\cosh t - 2\epsilon\cosh 2t)$.
For $\epsilon = -1$ **the second component never vanishes** ($\cosh t$ **always** ≤ 1); **for** $\epsilon = +1$ **we have**
first component $= 2\sinh t(2\cosh t + \epsilon)$, **which** $= 0$ **just when** $\sinh t = 0$, **i.e.** $t = 0$; **then** $\dot{\gamma}(0) = (0,0)$, $\gamma(0) = (3,0)$.
Write $\gamma_i(t) = (x_i(t), y_i(t))$ **for** $i = 1,2$ **and note the following facts**

- $x_i(t)$ **is an even function** ($\to +\infty$ **as** $t \to \pm\infty$), $y_i(t)$ **is an odd function.**
- $y_i(t) = 0$ **when** $2\sinh t = 2\epsilon\sinh t\cosh t$, **i.e.** $t = 0$ **or** $\epsilon\cosh t = 1$, **i.e.** $t = 0$ $(\epsilon = 1)$. **So** γ_1 **crosses the** x**-axis at** $(3,0)$, γ_2 **crosses the** x**-axis at** $(-1,0)$.
- $x_i(t) = 0$ **when** $\cosh 2t = -2\epsilon\cosh t$, **which does not happen when** $\epsilon = 1$.

When $\epsilon = -1$ **it occurs when** $c = \cosh t$ **satisfies** $2c^2 - 1 = 2c$, **i.e.** $c = \frac{1}{2}(1+\sqrt{3})$ (c **always** ≥ 1).
For $\epsilon = +1$ **we have**

$$\begin{aligned}\gamma(t) &= \left(1 + \frac{(2t)^2}{2!} + \cdots + 2\left(1 + \frac{t^2}{2!} + \cdots\right), 2\left(t + \frac{t^3}{3!} + \cdots\right) - \left(2t + \frac{(2t)^3}{3!} + \cdots\right)\right)\\ &= (3,0) + (3t^2 + \cdots, -t^3 + \cdots)\end{aligned}$$

$\Rightarrow \frac{3}{2}$**-power** **<u>cusp</u>.**
Also $\dot{x}_i(t)$ **has the same sign as** t, **and** $\dot{y}_i(t)$ **is always** ≤ 0 **for** $i = 1$ **and** > 0 **for** $i = 2$.
Finally, the curves must never meet, they lie on opposite sides of the curve

$$(\cosh 2t, \sinh 2t) = \gamma_0(t), \quad \textbf{i.e. } x^2 - y^2 = 1$$

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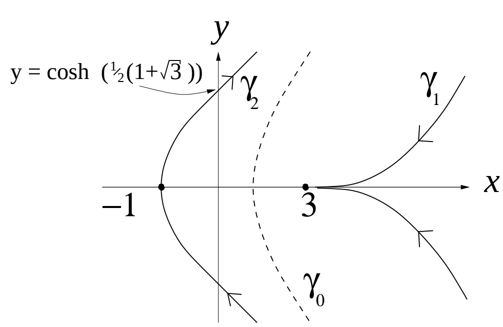

When $t = 0$ we find $\|\dot{\gamma}_2\| = 4$ and $\dot{x}\ddot{y} - \dot{y}\ddot{x} = -8$, so $\rho = \frac{1}{K} = -8$.
Also $N(0) = (-1, 0)$ so centre of curvature is

$$\gamma_2(0) + \rho N(0) = (7, 0).$$