Question

Solve the equation

$$y'' + (1 - \varepsilon x)y = 0,\ y(0) = 1,\ y'(0) = 0,\ \varepsilon \to 0^+$$

by regular perturbation theory up to order ε. By consideration of the relative size of the terms in the expansion, show that the regular perturbation ceases to be an accurate approximation for large value of x such that $x = O(\varepsilon^{-\frac{1}{2}})$. Show by substitution that when $x = O(\varepsilon^{-\frac{1}{2}})$, the solution then behaves like $y(x) = \cos\left(x - \dfrac{\varepsilon x^2}{4}\right) + O(\varepsilon^{-\frac{1}{2}})$. Deduce that the solution cannot be regular over the full range of $0 < x < +\infty$ as $\varepsilon \to 0^+$.

Answer

$y'' + (1 - \varepsilon x)y = 0;\ y(0) = 1,\ y'(0) = 0\ \ \varepsilon \to 0^+$

Try $y(x; \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$

Therefore $y_0'' + \varepsilon y_1'' + (1 - \varepsilon x)(y_0 + \varepsilon y_1) + O(\varepsilon^2) = 0$

$\underline{O(\varepsilon^0)}$: $y_0'' + y_0 = 0 \Rightarrow y_0 = A\sin x + B\cos x$

Boundary conditions $\Rightarrow$ $\begin{array}{rclcl} 1 &=& A\cdot 0 + B & \Rightarrow & B = 1 \\ 0 &=& A\cdot 1 + B\cdot 0 & \Rightarrow & A = 0 \end{array}$

Therefore $y_0 = \cos x$

$\underline{O(\varepsilon^1)}$: $y_1'' + y_1 - xy_0 = 0 \Rightarrow y_1'' + y_1 = x\cos x$

$y = CF + PI$

$y_{CF} = C\cos x + D\sin x$

<u>TRY</u> $y_{PI} = (\alpha^2 + \beta x)\cos x + (\phi x^2 + \gamma x)\sin x$

Substitution of y_{PI} in equation gives

$$2\alpha\cos x - 2(2\alpha x + \beta)\sin x + 2\phi\sin x + 2(2\phi x + \gamma)\cos x = x\cos x$$

Comparison of like terms gives

$$\alpha = 0,\ \beta = \frac{1}{4},\ \phi = \frac{1}{4},\ \gamma = 0$$

so $y_{PI} = \dfrac{1}{4}x\cos x + \dfrac{1}{4}x^2\sin x$

Therefore $y = C\cos x + \dfrac{1}{4}x\cos x + \dfrac{1}{4}x^2\sin x$.

Use boundary conditions to find C and D:

$$\left\{\begin{array}{ll} y_1(0) = 0 : & 0 = C \\ \underbrace{y_1'(0) = 0} : & 0 = \dfrac{1}{4} + D \end{array}\right\} \Rightarrow y_1 = \frac{1}{4}x^2\sin x + \frac{1}{4}x\cos x - \frac{1}{4}\sin x$$

from perturbation of boundary conditions $y'(0) = 0$.

Thus $y = \cos x + \varepsilon \left[\frac{1}{4}x^2 \sin x + \frac{1}{4}x \cos x - \frac{1}{4}\sin x\right] + O(\varepsilon^2)$

Clearly when $\varepsilon x^2 = O(1)$ the $\dfrac{\varepsilon x^2}{4}\sin x$ term amplitude has grown to be the same size as the $\cos x$, initial term. Thus the oscillations from the "small" ε correction are of the same size as the leading order behaviour. Hence the perturbation series breaks down here, i.e., where $x = O(\varepsilon^{-\frac{1}{2}})$. Therefore it can't be regular over $0 < x < +\infty$.

Given $y = \cos\left(x - \dfrac{\varepsilon x^2}{4}\right) + O(\varepsilon^{\frac{1}{2}})$,

$y' \sim -\left(1 - \dfrac{\varepsilon x}{2}\right)\sin\left(x - \dfrac{\varepsilon x^2}{4}\right)$

$y'' \sim \dfrac{\varepsilon}{2}\sin\left(x - \dfrac{\varepsilon x^2}{2}\right) - \left(1 - \dfrac{\varepsilon x}{2}\right)^2 \cos x$ and $y(0) = 1,\ y'(0) = 0 \Rightarrow y'' + y(1 - \varepsilon y) = O(\varepsilon)$ as $\varepsilon \to 0$

so leading order <u>uniform</u> expansion.