Question

Show that the eigenvalue problem

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y(1) + y'(1) = 0$$

has eigenvalues $\lambda = \mu^2$ with μ any root of $\mu \tan \mu = 1$. By means of a suitable sketch, show that this equation has a solution μ_n satisfying

$$n\pi < \mu_n < \left(n + \frac{1}{2}\right)\pi,\ n = 0, 1, 2, \cdots$$

Hence justify the approximation $\mu_n = n\pi + m_1$ where $m_1 = o(n\pi)$. Substitute this into $\mu \tan \mu = 1$ and expand in powers of m_1 showing that $m_1 = \dfrac{1}{n\pi}$.

Answer

$y'' + \lambda y = 0,\ y'(0) = 0\ \ (A),\ y(1) + y'(1) = 0\ \ (B)$
Clearly $y = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x\ \ A,\ B const$

Boundary conditions:
$(A): 0 = A\sqrt{\lambda}\cos 0 - B\sqrt{\lambda}\sin 0$
$(B): 0 = B\cos\sqrt{\lambda} - B\sqrt{\lambda}\sin\sqrt{\lambda}$
Assume $B \neq 0$ then must have

$$\cos\sqrt{\lambda} = \sqrt{\lambda}\sin\sqrt{\lambda} \Rightarrow \sqrt{\lambda}\tan\sqrt{\lambda} = 1$$

or if $\sqrt{\lambda} = \mu^2,\ \mu\tan\mu = 1$
PICTURE

From diagram (for μ not so large and positive) we have

$$n\pi < \mu_n < \left(n + \frac{1}{2}\right)\pi\ \ n \in \mathbf{Z}^+$$

Since root $\to 0$ in limit from above and $\tan\mu > 0$ for $n\pi < \mu < \left(n + \frac{1}{2}\right)\pi \;\; n \in \mathbf{Z}^+$.
Clearly root is small and $\tan\mu = 0$ when $\mu = n\pi$.
Therefore $\mu_n = n\pi + m_1$ where m_1 is small, say $o(n\pi)$
Substitute into $\mu\tan\mu = 1$:

$$
\begin{aligned}
&(n\pi + m_1)\tan(n\pi + m_1) = 1 \\
&(n\pi + m_1)\left[\frac{\tan n\pi + \tan m_1}{1 - \tan n\pi \tan m_1}\right] = 1 \\
&(n\pi + m_1)\tan m_1 = 1 \\
&\left(1 + \frac{m_1}{n\pi}\right)\tan m_1 = \frac{1}{n\pi}
\end{aligned}
$$

So to leading order in n

$$\tan m_1 = \frac{1}{n\pi}$$

As $n \to +\infty,\;\; m_1 \approx \tan m_1$ as m_1 is small $\Rightarrow m_1 \approx \frac{1}{n\pi}$
Therefore $\mu_n \sim n\pi + \frac{1}{n\pi} + o\left(\frac{1}{n\pi}\right)$.