Question

Find the general solution of the differential equation

$$y' + y^2 = 1$$

by separation of variables. Examine the same equation by dominant balance as $x \to +\infty$, comparing the results with the exact solution.

Answer

$y' + y^2 = 1 \Rightarrow y' = 1 - y^2 \Rightarrow \dfrac{y'}{1-y^2} = 1$

Therefore $\displaystyle\int \frac{dy}{1-y^2} = \int dx \Rightarrow y = \frac{1-Ae^{-2x}}{1+Ae^{-2x}} \quad A = const$

Dominant balance as $x \to +\infty$: Try

$\underline{y' = 1} \Rightarrow y = x + c \Rightarrow y^2 = O(x^2)$ so $y' = o(y^2)$. Inconsistent.

$\underline{y' = y^2} \Rightarrow \displaystyle\int \frac{y'}{y^2} = \int dx \Rightarrow -\frac{1}{y} = x + c \Rightarrow y = -\frac{1}{x+c} = o(1)$ as $x \to +\infty$.

Inconsistent.

$\underline{y^2 = 1} \Rightarrow y = \pm 1 \quad y' = 0 = o(1)$ as $x \to \infty$!

This is the balance.

Therefore $y \sim \pm 1$ as $x \to \infty$.

Second order balance: $y = +1 + y_1$ where $y_1 = o(1)$ $*$

(Take +1 only)

$$(1 + y_1)' + (1 + y_1)^2 = 1$$

Then $y_1' + 1 + 2y_1 + y_1^2 = 1$

$$y_1' + 2y_1 + y_1^2 = 0$$

<u>Balance</u>

$\underline{y_1' = -2y_1} \Rightarrow \displaystyle\int \frac{y_1'}{y'} = -2\int dx \Rightarrow y_1 = Be^{-2x} \Rightarrow y_1^2 = O(e^{-4x}) = o(e^{-2x}) \to$ <u>consistent</u>.

NB Other choice of -1 leads to inconsistency

$\to y \sim -1 \Rightarrow y_1 = O(e^{2x}) = o(e^{4x})$

Check others:

$\underline{y_1 = -y_1^2} \Rightarrow \displaystyle\int \frac{y_1'}{y_1^2} = -\int dx \Rightarrow -\frac{1}{y_1'} = -x + c \Rightarrow y_1 = O\left(\frac{1}{x}\right) \quad x \to +\infty.$

Hence $y_1^2 = o(y_1)$ as $x \to +\infty$. <u>INCONSISTENT</u>.

$\underline{2y_1 = -y_1^2} \Rightarrow y_1 = 0$ (gives an exact solution: $\underline{y = \pm 1} \Rightarrow y' = 0$ and $y^2 = 1$) or $y_1 = -2$. (This is not $o(1)$ as assumed $\star$)

Therefore we get either $y = \pm 1$ exactly or $y \sim 1 - Be^{-2x} \quad x \to +\infty$ which is consistent with the expansion of the exact result if $B = 2A$ (as $x \to +\infty$).