Question

A solution in descending powers of x is sought for the equation

$$y'' + y = \frac{1}{x}, \ x \to +\infty$$

Find the first few terms by direct substitution of the ansatz

$$y(x) \sim \sum_{r=0}^{\infty} \frac{a_r}{x^r}$$

Answer

$y'' + y = \dfrac{1}{x}$ (1) $x \to +\infty$

If $y \sim \sum_{r=0}^{\infty} \dfrac{a_r}{x^r}$ then $\begin{array}{l} y' \sim \sum_{r=0}^{\infty} -r\dfrac{a_r}{x^{r+1}} \\ y'' \sim \sum_{r=0}^{\infty} \dfrac{+r(r+1)a_r}{x^{r+2}} \end{array}$

Ignore Poincaré and differentiation of asymptotics. We're doing formal maths!

Hence in (1):

$\sum_{r=0}^{\infty} \dfrac{r(r+1)a_r}{x^{r+2}} + \displaystyle\sum_{r=0}^{\infty} \dfrac{a_r}{x^r} = \dfrac{1}{x}$

We balance at like powers of x.

$O(x^0): \quad a_0 = 0$

$O(x^{-1}): \quad a_1 = 1$

$O(x^{-2}): \quad a_2 = 0$

$O(x^{-3}): \quad 1 \cdot 2 \cdot a_1 + a_3 = 0 \quad \Rightarrow \quad a_3 = -2a_1 = -2$

$O(x^{-4}): \quad 2 \cdot 3 \cdot a_2 + a_4 = 0 \quad \Rightarrow \quad a_4 = -6a_2 = 0$

$O(x^{-5}): \quad 3 \cdot 4 \cdot a_3 + a_5 = 0 \quad \Rightarrow \quad a_5 = -12a_3 = +24$

Spot the pattern:

for $r = 2n, \ a_r = 0$

$r = 2n+1, \ a_{2n+1} = -(2n-1)(2n)a_{2n-1} \ \ (n \geq 1)$

$a_1 = 1 \ \ (n = 0)$

This will, in principle, determine all coefficients.

The series diverges (by ratio test) for all x.