Question

Use the method of matching to find the first terms in the outer and inner solutions of

$$\varepsilon y'' + (2x^2 + x + 1)y' = 4x + 1, \ y(0) = 1, \ y(1) = 1$$

given that a boundary layer of width $O(\varepsilon)$ exists near the origin. Hence write down the one-term composite expansion. Compare thus with the exact solution.

Answer

$\varepsilon y'' + (2x^2 + x + 1)y' = 4x + 1, \ y(0) = 1, \ y(1) = 1$

Assuming again an $O(\varepsilon)$ boundary layer near $x = 0$ we have

OUTER $y = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$

Substitute into equation

$$\varepsilon y_0'' + (2x^2 + x + 1)(y_0' + \varepsilon y_1') = 4x + 1 + O(\varepsilon^2)$$

$O(\varepsilon^0)$

$$\begin{aligned}
(2x^2 + x + 1)y_0' &= 4x + 1 \\
& \quad y_0(1) = 1 (isonlyrelevantboundarycondition) \\
\text{Therefore } y_0' &= \frac{4x+1}{2x^2 + x + 1} \\
\int dy_0 &= \int \frac{4x+1}{2x^2 + x + 1}\, dx \\
y_0 &= \int \frac{d(2x^2 + x + 1)}{2x^2 + x + 1} = A + \log(2x^2 + x + 1)
\end{aligned}$$

or see standard text books.

Apply boundary conditions to get

$$\begin{aligned}
1 &= \log(2 + 1 + 1) + A \\
\Rightarrow A &= 1 - \log 4 \\
\text{Therefore } y_0 &= \underline{1 + \log\left(\frac{2x^2 + x + 1}{4}\right)}
\end{aligned}$$

INNER

Assuming $O(\varepsilon)$ boundary layer near $x = 0$, set $\frac{x}{\varepsilon} = X \Rightarrow \partial_x = \frac{1}{\varepsilon}\partial_X$ with $y(\varepsilon X; \varepsilon) = Y(X; \varepsilon)$

Equation becomes

$\frac{1}{\varepsilon}Y'' + (2\varepsilon^2 X^2 + \varepsilon X + 1)Y' = 4\varepsilon^2 X + \varepsilon \ \ Y(0) = 1$

Only 1 boundary condition for 2nd order equation $\Rightarrow$ matching needed for finding 2nd arbitrary constant.
Solve perturbatively using regular constants
$Y(X,\varepsilon) = Y_0(X) + \varepsilon Y_1(\varepsilon) + \varepsilon^2 Y_2(\varepsilon) + O(\varepsilon^3)$
$\underline{O(\varepsilon^0)} Y_0'' + Y_0' = 0,\ Y_0(0) = 1$
This is the same as in question 8.
Hence solution:

$$Y_0 = (1-c) + ce^{-X}$$

c determined from matching by Van Dyke.

$$\begin{aligned}
\text{One term outer expansion} &= 1 + \log\left(\frac{2x^2+x+1}{4}\right) \\
\text{Rewritten in inner variable} &= 1 + \log\left(\frac{2\varepsilon^2X^2+\varepsilon X+1}{4}\right) \\
\text{Expanded for small } \varepsilon &= 1 - \log 4 + \varepsilon X + O(\varepsilon^2) \\
\text{One term } O(\varepsilon^0) &= 1 - \log 4 \ (\star) \\
\text{One term inner expansion} &= ce^{-X} - (c-1) \\
\text{Rewritten in outer variable} &= ce^{-\frac{x}{\varepsilon}} - (c-1) \\
\text{Expanded for small } \varepsilon &= 1 - c \\
& \quad + \text{exp. small in} \varepsilon \\
\text{One term } O(\varepsilon^0) &= 1 - c \ (\star\star)
\end{aligned}$$

$(\star) and (\star\star)$ must match, hence: $1 - c = 1 - \log 4 \Rightarrow c = \log 4$
Therefore outer 1-term is: $y(x;\varepsilon) = 1 + \log\left(\frac{2x^2+x+1}{4}\right) + O(\varepsilon)$
Inner 1-term is $Y(X;\varepsilon) = (1-\log 4) + (\log 4)e^{-X}$
Composite is:

$$\begin{aligned}
y^{comp} &= y^{outer} + y^{inner} - \text{inner limit of } y^{outer} \\
&= 1 + \log\left(\frac{2x^2+x+1}{4}\right) + (1-\log 4) \\
&\quad + (\log 4)e^{-X} - (1-\log 4) \\
y^{comp} &= \underline{1 + \log\left(\frac{2x^2+x+1}{4}\right) + (\log 4)e^{-\frac{x}{\varepsilon}} + \cdots} \quad \varepsilon \to 0^+
\end{aligned}$$

Exact solution is difficult to find!