Question

Use the method of matching to find the first terms in the outer and inner solutions of

$$\varepsilon y'' + y' = 1,\ y(0) = \alpha,\ y(1) = \beta.$$

given that a boundary layer of width $O(\varepsilon)$ exists near the origin. Hence write down the one-term composite expansion. Compare thus with the exact solution.

Answer

$\varepsilon y'' + y' = 1,\ y(0) = \alpha,\ y(1) = \beta$

Assume that the boundary layer exists near $x = 0$ of width $O(\varepsilon)$ $\leftarrow$ This needs to be justified really but we follow lead from the question.

<u>OUTER</u> $y = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$

$$\varepsilon y_0'' + y_0' + \varepsilon y_1' = 1 + O(\varepsilon^2)$$

Balance at

$\underline{O(\varepsilon^0)}: y_0' = 1 \Rightarrow y_0 = x + A$

Boundary data: only relevant one is $y(1) = \beta$. (0 is in the boundary layer).

$$\Rightarrow y_0(1) = \beta \text{ so } \underline{\mathrm{y_0 = \beta - 1 + x}}$$

<u>INNER</u>

Assuming $O(\varepsilon)$ boundary layer near $x = 0$, set $X = \dfrac{x}{\varepsilon}$ as inner variable.

$\partial_x = \dfrac{\partial X}{\partial x}\partial_X = \dfrac{1}{\varepsilon}\partial_X$ etc. with $y(\varepsilon X; \varepsilon) = Y(X, \varepsilon)$

Equation becomes:

$\dfrac{1}{\varepsilon}Y''(X, \varepsilon) + \dfrac{1}{\varepsilon}Y'(X, \varepsilon) = 1$

Therefore $Y'' + Y' - \varepsilon = 0$ with $Y(0) = \alpha$

2nd order equation and only one boundary condition $\Rightarrow$ matching is needed to find 2nd arbitrary const.

Try regular ansatz: $Y(X; \varepsilon) = Y_0(X) + \varepsilon Y_1(\varepsilon) + O(\varepsilon^2)$

$\underline{O(\varepsilon^0)} Y_0'' + Y_0' = 0;\ Y_0(0) = \alpha$

$\rightarrow Y_0^(X) = A + Ce^{-X}$ where A and C are arbitrary constants.

Therefore $\alpha = A + C$, can only find 1 constant. Pick A say.

$A = \alpha - c$

Therefore $Y_0(X) = (\alpha - c) + ce^{-X}$

Match up to get value of c using Van Dyke.

One term outer expansion	$=$	$\beta - 1 + x$
Rewritten in inner variable	$=$	$\beta - 1 + \varepsilon x$
Expanded for $\varepsilon \to o^+$	$=$	$\beta - 1 + \varepsilon x$
One term $O(\varepsilon^0)$	$=$	$\beta - 1$ $(\star)$
One term inner expansion	$=$	$\alpha - c + ce^{-X}$
Rewritten in outer variable	$=$	$\alpha - c + ce^{-\frac{x}{\varepsilon}}$
Expanded for $\varepsilon \to 0^+$	$=$	$\alpha - c$
		+exp. small term inε
One term $O(\varepsilon^0)$	$=$	$\alpha - c$ $(\star\star)$

$(\star) and (\star\star)$ must be equal:
$\Rightarrow \beta - 1 + \alpha - c$ or $\underline{c = \alpha - \beta + 1}$
Therefore outer 1-term is: $y(x;\varepsilon) = \beta - 1 + x$
Inner 1-term is $Y(X;\varepsilon) = (\alpha - \beta + 1)e^{-\frac{x}{\varepsilon}} + (\beta - 1)$
Composite is:

$$\begin{aligned} y^{comp} &= y^{outer} + y^{inner} - \text{inner limit of } y^{outer} \\ &= \beta - 1 + x + (\alpha - \beta + 1)e^{-\frac{x}{\varepsilon}} + (\beta - 1) - (\beta - 1) + \cdots \\ &= (\beta - 1) + x + (\alpha - \beta + 1)e^{-\frac{x}{\varepsilon}} + \cdots, \quad \varepsilon \to 0^+ \end{aligned}$$

Exact solution is given by $y = \underbrace{y^{CF}}_{(1)} + \underbrace{y^{PI}}_{(2)}$

(1) solves $\varepsilon y'' + y' = 0$
(2) PI of $\varepsilon y'' + y' = 1 \underline{y^{PI} = x}$
After using boundary conditions we get

$$y = \frac{x + (\beta - 1) - \alpha e^{-\frac{1}{\varepsilon}} + (\alpha - \beta + 1)e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} \ \underline{\text{exactly}}$$

so as $\varepsilon \to 0^+$, $e^{-\frac{1}{\varepsilon}}$ is exp. small with respect to $e^{-\frac{x}{\varepsilon}}$ $x \in [0, 1)$
so $y \sim x + (\beta - 1) + (x - \beta + 1)e^{-\frac{x}{\varepsilon}} +$ exp. small terms $\surd\surd$