Question

Obtain the first two terms of an asymptotic expansion for solutions of the following differential equations as $x \to +\infty$ using the method of dominant balance.

(a) $y' + y^4 = x$

(b) $y' + y^{\frac{1}{2}} = x^2$

Answer

(a) $\underset{(1)}{\underset{\uparrow}{y'}} + \underset{(2)}{\underset{\uparrow}{y^4}} = \underset{(3)}{\underset{\uparrow}{x}}$

Look for dominant balance solution as $x \to +\infty$.

<u>(1) and (3)</u>

$y' = x \Rightarrow y = \dfrac{x^2}{2} + c$

But then $y^4 = O(x^8)$ as $x \to +\infty$ which is <u>inconsistent</u> as we've assumed $y^4 = o(y')$ and $o(x)$.

<u>(1) and (2)</u>

$y' = -y^4 \Rightarrow \displaystyle\int \frac{dy}{y^4} = -\int dx \Rightarrow -\frac{1}{3y^3} = -x + c$

$\Rightarrow y^3 = \dfrac{1}{3(x+c)} \Rightarrow y = \dfrac{1}{[3(x+c)]^{\frac{1}{3}}} = o(x)$ as $x \to +\infty$ Therefore <u>inconsistent</u>.

<u>(2) and (3)</u>

$y^4 = x \Rightarrow y = \pm x^{\frac{1}{4}} \Rightarrow y' = O(x^{-\frac{3}{4}})$ as $x \to +\infty$ so $y' = o(y^4)$ and $o(x)$ $\surd\surd$

This is a good balance. Hence $y \sim \pm x^{\frac{1}{4}},\ \pm i x^{\frac{1}{4}}$ as $x \to +\infty$ (boundary conditions would determine)

(b) $\underset{(1)}{\underset{\uparrow}{y'}} + \underset{(2)}{\underset{\uparrow}{y^{\frac{1}{2}}}} = \underset{(3)}{\underset{\uparrow}{x^2}}$

Look for dominant balance solutions as $x \to +\infty$.

<u>(1) and (3)</u>

$y' = x^2 \Rightarrow y = \dfrac{x^3}{3} + c \Rightarrow y^{\frac{1}{2}} = O(x^{\frac{3}{2}}) \quad x \to +\infty$

and $x^{\frac{3}{2}} = o(x^2)$ so this <u>looks</u> consistent.

We must still check the other possible balances to make sure there's no ambiguity.

<u>(1) and (2)</u>

$y' + y^{\frac{1}{2}} = 0 \Rightarrow \int \dfrac{dy}{\sqrt{y}} = -\int dx \Rightarrow 2y^{\frac{1}{2}} = -x + c \Rightarrow y = \dfrac{(x-c)^2}{4}$

Therefore $y^{\frac{1}{2}} = O(x) \quad x \to +\infty$ but therefore

$y^{\frac{1}{2}} = o(x^2)$ an inconsistency.

<u>(2) and (3)</u>

$y^{\frac{1}{2}} = x^2 \Rightarrow y = x^4 \Rightarrow y' = 4x^3$ so $y^{\frac{1}{2}} = o(y')$ an inconsistency.

Hence $y \sim \dfrac{x^3}{3} + c$ as x $\to +\infty$.