Question
Prove that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges for $s < 1$, by estimating its partial sums.

Answer
Note that for $s < 1$, we have that $n^s < n$ (even for $s = 0$ or s negative), and hence that $\frac{1}{n^s} > \frac{1}{n}$. Hence, if we let S_k be the k^{th} partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, and T_k be the k^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ under consideration, then $T_k > S_k$. Since $\frac{1}{n^s} > 0$ for all n, we have that $\{T_k\}$ is an unbounded monotonically increasing sequence, unbounded since $\{S_k\}$ is unbounded by the argument given below, and so $\{T_k\}$ diverges. So, by definition, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges.

Argument
The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.

For $s = 1$, this series is called the **harmonic series**, and we can prove directly that it diverges. Note that $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$, that $\frac{1}{5} + \cdots + \frac{1}{8} > 4\frac{1}{8} = \frac{1}{2}$, and in general that

$$\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 2^{k-1}\frac{1}{2^k} = \frac{1}{2}.$$

Hence, the $(2^k)^{th}$ partial sum S_{2^k} satisfies $S_{2^k} > 1 + k\frac{1}{2}$. Since the terms in the harmonic series are all positive, the sequence of partial sums is monotonically increasing, and by the calculation done the sequence of partial sums is unbounded, and so the sequence of partial sums diverges. Hence, the harmonic series diverges.