Question

1. Show that, if $\sum_{n=0}^{\infty} a_n$ converges and if $\sum_{n=0}^{\infty} b_n$ diverges, then the series of sums $\sum_{n=0}^{\infty}(a_n + b_n)$ diverges.

2. Show that, if $\sum_{n=0}^{\infty} a_n$ diverges and if $c \neq 0$, then the series of multiples $\sum_{n=0}^{\infty} c\, a_n$ diverges.

Answer

1. we argue by contradiction: suppose that $\sum_{n=0}^{\infty}(a_n+b_n)$ converges. Since $\sum_{n=0}^{\infty} a_n$ converges by assumption, the arithmetic series, yields that their difference also converges. However, their difference is $\sum_{n=0}^{\infty}(a_n + b_n - a_n) = \sum_{n=0}^{\infty} b_n$, which diverges by assumption, yielding the desired contradiction.

2. again we argue by contradiction: suppose that the series of multiples $\sum_{n=0}^{\infty} ca_n$ converges. Then, the sequence $\{T_k = \sum_{n=0}^{k} ca_n\}$ of partial sums converges. Note though that $T_k = \sum_{n=0}^{k} ca_n = c\sum_{n=0}^{k} a_n = cS_k$, where S_k is the k^{th} partial sum of the series $\sum_{n=0}^{\infty} a_n$. Since $\{T_k\}$ converges, the sequence $\{\frac{1}{c}T_k = S_k\}$ also converges, by the arithmetic of sequences (since the constant sequence $\{\frac{1}{c}\}$ converges), and so the original series converges, a contradiction.