Exam Question
Topic: CriticalPoints
Find and classify the critical points of the function

$$f(x, y) = y(x - 3)^2 - (x - y)(x + y - 6).$$

Solution
$f(x, y) = y(x - 3)^2 - (x - y)(x + y - 6)$, so equating the partial derivatives to zero gives

$$\begin{aligned} f_x &= 2y(x - 3) - 2x + 6 = 0 & (1) \\ f_y &= (x - 3)^2 + 2y - 6 = 0 & (2) \end{aligned}$$

Equation (1) gives $2(x - 3)(y - 1) = 0$ so $x = 3$ or $y = 1$. From equation (2), $x = 3$ gives $2y - 6 = 0$; $y = 3$.
Also from equation (2), $y = 1$ gives $(x - 3)^2 = 4$, so $x = 1$ or $x = 5$.
So the critical points are $(1, 1)$, $(3, 3)$, $(5, 1)$.
the second partial derivatives are $f_{xx} = 2y - 2$, $f_{yy} = 2$, $f_{xy} = 2(x - 3)$.

	f_{xx}	f_{yy}	f_{xy}	$\Delta = f_{xy}^2 - f_{xx}f_{yy}$	Type
$(1, 1)$	0	2	-4	$16 > 0$	SADDLE
$(3, 3)$	4	2	0	$-8 < 0$	MIN ($f_{xx} > 0$)
$(5, 1)$	0	2	4	$16 > 0$	SADDLE

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