In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question Find the critical curves of the functionals

(i) $\displaystyle\int_a^b (y^2 + y'^2 - 2y\sin x)\,dx$

(ii) $\displaystyle\int_a^b (y^2 - y'^2 - 2y\sin x)\,dx$

(iii) $\displaystyle\int_a^b (y^2 - y'^2 - 2y\cosh x)\,dx$

(iv) $\displaystyle\int_a^b (y^2 + y'^2 - 2ye^x)\,dx$

Answer

(i) $F(y, y', x) = y^2 + y'^2 - 2y\sin x$

$\dfrac{\partial F}{\partial y} = 2y - 2\sin x;\ \dfrac{\partial F}{\partial y'} = 2y'$

E-L equation becomes:

$$\begin{aligned} (2y - 2\sin x) - \frac{d}{dx}(2y') &= 0 \\ \Rightarrow y - \sin x - y'' &= 0 \\ \Rightarrow \qquad y'' - y &= -\sin x \end{aligned}$$

Inhomogeneous 2nd order linear equation ODE with constant coefficients.

So $y = y_{comp.func.} + y_{partic.int.}$

$y_{cf} Ae^{mx} + Be^{-mx}$ where A and B are constants

where by substitution

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

Therefore $y_{cf} = Ae^x + Be^{-x}$

For particular integral try

$y_{PI} = C\cos x + D\sin x$

$y'_{PI} = -C\sin x + D\cos x$

$y''_{PI} = -C\cos x - D\sin x$

Substitution in (1) gives

$-C\cos x - D\sin x - C\cos x - D\sin x = -\sin x$

$\Rightarrow C = 0,\ D = \frac{1}{2}$

Therefore $t = Ae^x + Be^{-x} + \frac{1}{2}\sin x$ is extremal function would need to find A and B using boundary data; but we haven't been given any (only that $x = a$ and $x = b$ are the end points)

(ii) $F(y, y', x) = y^2 - y'^2 - 2y\sin x$

$\frac{\partial F}{\partial y} = 2y - 2\sin x;\ \frac{\partial F}{\partial y'} = -2y'$

E-L equation becomes:

$$\begin{aligned}(2y - 2\sin x) - \frac{d}{dx}(-2y') &= 0\\ \Rightarrow y - \sin x + y'' &= 0 \qquad (2)\\ \Rightarrow \quad y'' + y &= \sin x\end{aligned}$$

Same type of equation as in (i). Use same method to get solution

$y = -\frac{1}{2}x\cos x + A\cos x + B\sin x$

A and B to be determined from boundary data.

(iii) $F(y, y', x) = y^2 - y'^2 - 2y\cosh x$

$\frac{\partial F}{\partial y} = 2y - 2\cosh x;\ \frac{\partial F}{\partial y'} = -2y'$

E-L equation becomes:

$$\begin{aligned}(2y - 2\cosh x) - \frac{d}{dx}(-2y') &= 0\\ \Rightarrow y - \cosh x + y'' &= 0 \qquad (2)\\ \Rightarrow \quad y'' + y &= \cosh x\end{aligned}$$

Same type of equation as above. Use similar method to get solution

$y = -\frac{1}{2}\cosh x + A\cos x + B\sin x$

A and B to be determined from boundary data.

(iv) $F(y, y', x) = y^2 + y'^2 + 2ye^x$

$\frac{\partial F}{\partial y} = 2y + 2e^x; \ \frac{\partial F}{\partial y'} = 2y'$

E-L equation becomes:

$$\begin{aligned} (2y + 2e^x) - \frac{d}{dx}(2y') &= 0 \\ \Rightarrow y + e^x - y'' &= 0 \\ \Rightarrow \qquad y'' - y &= e^x \end{aligned}$$

Same type of equation as above. Use similar methods to get solution

$y = -\frac{1}{2}xe^x + Ae^x + Be^{-x}$

A and B to be determined from boundary data.