In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

Find the curve joining the points $(0, \sqrt{2})$ and $(1, 1)$ which makes

$$I = \int_0^1 dx y^{-1}(1 + y'^2)^{\frac{1}{2}}$$

stationary, and show that for this curve $I = arctanh\left(\frac{1}{\sqrt{2}}\right)$.

Answer

Again $F = \dfrac{\sqrt{1+y'^2}}{y} = F(y, y')$ only, so $y'\dfrac{\partial F}{\partial y'} - F = const$

Therefore $\dfrac{\partial F}{\partial y'} = \dfrac{1}{2}\dfrac{(1+y'^2)}{y}^{0-\frac{1}{2}} \times 2y' = \dfrac{y'}{y\sqrt{1+y'^2}}$

Therefore $\dfrac{y'}{y\sqrt{1+y'^2}} - \dfrac{\sqrt{1+y'^2}}{y} = const$

Therefore $\dfrac{1}{y\sqrt{1+y'^2}} = const = \alpha$ say

$\Rightarrow y' = \pm\left(\dfrac{1}{\alpha^2 y^2} - 1\right)^{\frac{1}{2}}$

$\Rightarrow \alpha x + c = \pm(1 - \alpha^2 y^2)^{\frac{1}{2}}$ (standard integration)

$\Rightarrow (\alpha x + c)^2 = 1 - \alpha^2 y^2$

$y(0) = \sqrt{2},\ y(1) = 1 \Rightarrow c = 0, \alpha^2 = \dfrac{1}{2}$ and so extremals is the circle $\underline{x^2 + y^2 = 2}$

In this case, $I = \displaystyle\int_0^1 \frac{1}{y}\left(x + \frac{x^2}{y^2}\right)^{\frac{1}{2}} dx$

$= \displaystyle\int_0^1 \frac{\sqrt{2}\,dx}{2 - x^2} = \underline{arctanh\left(\frac{1}{\sqrt{2}}\right)}$ standard integral