In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

Find $y(x)$ such that

$$I = \int_0^a dx(y^2 + 2y'^2 + y''^2)$$

is stationary and $y(0) = 1 + \frac{5a}{4},\ y(a) = \frac{5}{3} + \frac{3a}{4},\ y'(0) = 0,\ y'(a) = 0$ where $a = \log 3$.

Answer

The relevant E-L equation is (generalistion 1)

$$\frac{d^2}{dx^2}(2y'') - \frac{d}{dx}(4y') + 2y = 0$$

which gives

$$y^{iv} - 2y'' + y = 0$$

From lectures it is convenient to write the solution as

$$y = A\cosh x + B\sinh x + Cx\cosh x + Dx\sinh x$$

$y'(0) = 0 \Rightarrow 0 = B + C$

$y(0) = 1 + \frac{5\log 3}{4} \Rightarrow A = 1 + \frac{5}{4}\log 3$

Since at $x = \log 3,\ \cosh x = \frac{5}{3},\ \sinh x = \frac{4}{3}$ we have that the 2 remaining conditions $\Rightarrow B = 0$ (and so $C = 0$), and $D = -1$. Hence $y = A\cosh x - x\sinh x$, where $A = 1 + \frac{5}{4}\log 3$.