Question

The equation governing the temperature $u(x,t)$ in a bar of metal of length l is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{k}\frac{\partial u}{\partial t} = 0$$

The ends of the bar are held at zero temperature, so that

$$u(0,t) = 0 \text{ and } u(l,t) = 0 \text{ for all } t > 0$$

The initial temperature is given by $u(x,0) = f(x)$ for $0 \le x \le l$.

(a) Find the temperature $u(x,t)$ at some future time t if $f(x)$ is given by

$$f(x) = 3\sin\left(\frac{4\pi x}{l}\right)$$

(b) Find the temperature if $k = 100$, $l = 1$ and $f(x)$ is given by

$$f(x) = \sin 2\pi x - 2\sin 5\pi x \qquad 0 \le x \le 1$$

Answer

Let $y(x,t) = X(x)T(t)$ so that

$$\frac{X''}{X} = \frac{1}{k}\frac{T''}{T} = \lambda$$

$$\Rightarrow X'' - \lambda X = 0$$

$X(0) = 0$ and $X(l) = 0$

$$\Rightarrow \lambda = -\frac{n^2\pi^2}{l^2}$$

Therefore

$$\begin{aligned} X_n(x) &= A_n \sin\left(\frac{n\pi x}{l}\right) \\ T' &= -\frac{n^2\pi^2 k}{l^2}T \\ T_n(t) &= B_n e^{-\frac{n^2\pi^2 kt}{l^2}} \end{aligned}$$

Thus

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}} \\
f(x) &= \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right)
\end{aligned}
$$

If $f(x) = 3\sin\left(\frac{4\pi x}{l}\right)$ then $Q_4 = 3$ and $Q_n = 0$ otherwise. So

$$u(x,t) = 3\sin\left(\frac{4\pi x}{l}\right) e^{-\frac{16\pi^2 kt}{l^2}}$$

If $l = 1$ then $f(x) = \sum_{n-1}^{\infty} Q_n \sin(n\pi x)$. But $f(x) = \sin(2\pi x) - 2\sin(5\pi x)$
$\Rightarrow Q_2 = 1 \;\; Q_5 = -2$ and $Q_n = 0$ otherwise. So

$$u(x,t) = \sin(2\pi x)e^{-400\pi^2 t} - 2\sin(5\pi x)e^{-2500\pi^2 t}$$