Question
Use the method of variation of parameters to find the general solution to each of the following equations.

(a) $y'' - 2y' + y = 4e^x$

(b) $y'' - 2y' + 2y = 4e^x \sin x$

(c) $y'' - 4y' + 4y = xe^{2x}$

Answer

(a)

$$\begin{aligned} y'' - 2y' + y &= 0 \\ \text{A.E.}\ \ m^2 - 2m + 1 &= 0 \\ (m-1)^2 &= 0 \\ m = 1 \ \ \text{twice} & \\ \text{so}\ \ y_1 = e^x \quad \text{and} \quad y_2 = xe^x & \end{aligned}$$

are the solutions of the homogeneous equation.

Looking at the Wronskian:

$W = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix} = e^{2x}$

The general solution has the form $y = v_1 y_1 + v_2 y_2$ where:

$v_1' = -\dfrac{y_2 R}{W} = -4x \Rightarrow v_1 = -2x^2 + A$

$v_2' = \dfrac{y_1 R}{W} = 4 \Rightarrow v_2 = 4x + B$

Hence

$$y = (A + B + 2x^2)e^x$$

(b)

$$\begin{aligned} y'' - 2y' + 2y &= 0 \\ \text{A.E.}\ \ m^2 - 2m + 2 &= 0 \\ m = 1 \pm i & \\ \text{so}\ \ y_1 = e^x \cos x \quad \text{and} \quad y_2 = e^x \sin x & \end{aligned}$$

are the solutions of the homogeneous equation.

Looking at the Wronskian:

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix} = e^{2x}$$

The general solution has the form $y = v_1 y_1 + v_2 y_2$ where:

$v_1' = -\dfrac{y_2 R}{W} = -4\sin^2 x = 2\cos^2 2x - 2 \Rightarrow v_1 = \sin 2x - 2x + A$

$v_2' = \dfrac{y_1 R}{W} = 4\sin x \cos x = 2\sin 2x \Rightarrow v_2 = -\cos x + B$

Hence

$$\begin{aligned} y &= (\sin 2x - 2x + A)e^x \cos x + (-\cos 2x + B)e^x \sin x \\ &= (\sin 2x \cos x - \cos 2x \sin x)e^x - 2x\cos x e^x \\ &\quad +(A\cos x + B\sin x)e^x \\ &= [(A\cos x + C\sin x) - 2x\cos x]e^x \end{aligned}$$

(c)

$$\begin{aligned} y'' - 4y' + 4y &= 0 \\ \text{A.E.} \quad m^2 - 4m + 4 &= 0 \\ (m-2) &= 0 \\ m = 2 \quad &\text{twice} \\ \text{so} \quad y_1 = e^{2x} \quad &\text{and} \quad y_2 = xe^{2x} \end{aligned}$$

are the solutions of the homogeneous equation.

Looking at the Wronskian:

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x}$$

The general solution has the form $y = v_1 y_1 + v_2 y_2$ where:

$v_1' = -\dfrac{y_2 R}{W} = -x^2 \Rightarrow v_1 = -\dfrac{1}{3}x^3 + A$

$v_2' = \dfrac{y_1 R}{W} = x \Rightarrow v_2 = \dfrac{1}{2}x^2 + B$

Hence

$$\begin{aligned} y &= \left(-\frac{1}{3}x^3 + A\right)e^{2x} + \left(\frac{1}{2}x^2 + B\right)xe^{2x} \\ &= \left(A + Bx + \frac{1}{6}x^3\right)e^{2x} \end{aligned}$$