Question
Use the method of reduction of order to obtain a second solution to each of the following equations using the given solution y_1

(a) $y'' - 4y' + 5y = 0, \qquad y_1 = e^{2x}\sin x,$

(b) $(x^2+1)y'' + 2xy' = 0, \qquad y_1 = c$, a constant,

(c) $x^3y''' + 3x^2y'' + xy' - 8y = 0, \qquad y_1 = x^2.$

Answer

(a) $y'' - 4y' + 5y = 0, \qquad y_1 = e^{2x}\sin x$

$P(x) = -4$ so $\exp\int -P dx = e^{4x}$

and $\dfrac{1}{y_1^2}\exp\int -P dx = \dfrac{1}{\sin^2 x}$

$$\begin{aligned} v &= \int \frac{1}{e^{4x}\sin^2 x}\cdot e^{4x}\,dx \\ &= \int \frac{dx}{\sin^2 x} \\ &= \csc^2 x\,dx \\ &= -\cot x \end{aligned}$$

Therefore $y_2 = vy_1 = -e^{2x}\cos x$ is a second solution.

The General solution is

$$y = -e^{2x}(A\cos x + B\sin x).$$

(b) $(x^2+1)y'' + 2xy' = 0, \qquad y_1 = c$, a constant,

Therefore $P(x) = \dfrac{2x}{x^2+1}$ so

$$\begin{aligned} \exp\int -P\,dx &= \exp\int -\frac{2x}{x^2+1}dx \\ &= \exp(-\ln(x^2+1)) \\ &= \frac{1}{x^2+1} \end{aligned}$$

Thus $\dfrac{1}{y_1^2}\exp\int -P\,dx = \dfrac{1}{x^2+1}$ taking (c = 1)

and $v = \displaystyle\int \frac{dx}{x^2+1} = \arctan x$

So $y_2 = vy_1 = \arctan x$ is a second solution.

The general solution is

$$y = A + B\,\arctan x$$

(c) $x^3y''' + 3x^2y'' + xy' - 8y = 0 \qquad (1)$

Let $y_1 = x^2$ and $y = vy_1 = x^2v$

$$\begin{aligned} y' &= 2xv + x^2v' \\ y'' &= 2v + 4xv' + x^2v'' \\ y''' &= 6v' + 6xv'' + x^2v''' \end{aligned}$$

Substituting in (1) gives

$$x^5v''' + 9x^4v'' + 19x^3v' = 0$$

Let $w = v'$ then

$$x^2w'' + 9xw' + 19w = 0 \qquad (2)$$

This is an Euler type equation.

Try $w = x^p$. Substituting in (2) gives

$$\begin{aligned} (p(p-1) + 9p + 19) &= 0 \\ \Rightarrow p^2 + 8p + 19 &= 0 \\ \Rightarrow p &= -4 \pm \sqrt{3}i \\ w &= x^{-4\pm\sqrt{3}i} \\ \Rightarrow v' &= x^{-4\pm\sqrt{3}i} \\ \Rightarrow v &= Cx^{-3\pm\sqrt{3}i} \end{aligned}$$

Now:

$$\begin{aligned} x^{-3+\sqrt{3}i} = x^{-3}x^{\sqrt{3}i} &= x^{-3}e^{\sqrt{3}i\ln x} \\ &= x^{-3}(\cos(\sqrt{3}\ln x) + i\sin(\sqrt{3}\ln x)) \end{aligned}$$

Hence the real v is given by

$v = x^{-3}\left(A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)\right)$
and $y_2 = y_1 v = x^2 v = x^{-1}\left(A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)\right)$
So the general solution is

$$y = x^{-1}\left(A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)\right) + Cx^2$$