Question
Describe what is meant by a Compound Poisson process.

Show that if $A(z)$ is the probability generating function for the number of events occurring at each point of the process, then the random variable $X(t)$ - the total number of events occurring in time t - has a probability generating function of the form

$$G(z) = \exp(\lambda t A(z) - \lambda t)$$

During the working day (8 a.m. to 6 p.m.) the Highfield Patent Medicine Co. receives telephone calls ordering various numbers of bottles of Dr. Hirst's Rejuvenating Elixir. The telephone calls arrive according to a Poisson process with rate λ calls per hour. The number N of bottles ordered by a telephone call has a geometric distribution, i.e.

$$P(N = n) = p(1-p)^{n-1}, \quad n = 1, 2, \cdots$$

Find the mean and variance of the number of bottles ordered per day.

Answer
Suppose that

(i) points occur in a Poisson process $\{N(t) : t \geq 0\}$ with rate λ

(ii) at the ith point Y_i event occur, where $Y_1, Y_2, ...$ are i.i.d. random variables's

(iii) Y_i and $\{N(t) : t \geq 0\}$ are independent. The total number of events occurring in a time interval of length t i s

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

$\{N(t) : t \geq 0\}$ is said to be a compound Poisson process.
Let the p.g.f. of each Y_i be A(z). Then X(t) has p.g.f.

$$\begin{aligned}\sum_{j=0}^{\infty} z^j P(X(t) = j) &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} z^j P(X(t) = j|N(t) = n)P(N(t) = n) \\ &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} z^j P(Y_1 + Y_2 + ... + Y_n = j)\frac{\lambda t^n e^{-\lambda t}}{n!}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} z^j P(Y_1 + Y_2 + ... + Y_n = j) \right\} \frac{\lambda t^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} [A(n)]^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&\quad \text{since the } Y_i \text{ are independent} \\
&= \exp(\lambda t A(z) - \lambda t) \\
&= G(z)
\end{aligned}
$$

For the geometric distribution

$$
\begin{aligned}
A(z) &= \sum_{n=1}^{a} inftyp(1-p)^{n-1} z^n \\
&= pz \sum_{n=1}^{\infty} (1-p)^{n-1} z^{n-1} \\
&= \frac{pz}{1-(1-p)z}
\end{aligned}
$$

From 8a.m. to 6p.m. there are 10 hours. So the p.g.f. for the number of bottles ordered are day is

$$\begin{aligned}
G(z) &= \exp\left(\frac{10\lambda pz}{1-(1-p)z} - 10\lambda\right) \\
G'(z) &= \exp\left(\frac{10\lambda pz}{1-(1-p)z} - 10\lambda\right) \\
&\quad \cdot\frac{(1-(1-p)z)+z(1-p)}{(1-(1-p)z)^2}\cdot 10\lambda p \\
&= \exp\left(\frac{10\lambda pz}{1-(1-p)z} - 10\lambda\right)\cdot\frac{10\lambda p}{(1-(1-p)z)^2} \\
G''(z) &= \exp\left(\frac{10\lambda pz}{1-(1-p)z} - 10\lambda\right)\left[\frac{10\lambda p}{(1-(1-p)z)^2}\right] \\
&\quad +\exp\left(\frac{10\lambda pz}{1-(1-p)z} - 10\lambda\right)\cdot\frac{2(1-p)\cdot 10\lambda p}{(1-(1-p)z)^3} \\
G'(1) &= \exp(0)\cdot\frac{10\lambda p}{p^2} \\
\text{So }\ E(X) &= \frac{10\lambda}{p} \\
G''(1) &= \left(\frac{10\lambda}{p}\right)^2 + \frac{20\lambda(1-p)}{p^2} \\
Var(X) &= G''(1)+G'(1)-G'(1)^2 \\
&= \frac{20\lambda(1-p)}{p^2}+\frac{10\lambda}{p} \\
&= \frac{20\lambda}{p^2}-\frac{\lambda}{p} \\
&= \frac{10\lambda}{p^2}(2-p)
\end{aligned}$$