Question
Explain what a branching Markov chain is. Suppose such a Markov chain begins with single individual. Let $A(s)$ denote the probability generating function for the number of offspring of any individual. State how to use $A(s)$ to find the probability of extinction. Prove that extinction occurs with probability 1 if and only if the mean number of offspring per individual does not exceed 1.

Let $F_n(s)$ denote the probability generating function for the number of individuals in generation n. Assuming the relationship $F_n(s) = F_{n-1}(s)(A(s))$, obtain expressions for the mean and variance of the number of individuals in generation n, in terms of the mean and variance of the number of offspring of any individual.

An organism reproduces by multiple division. The number of offspring of any individual has a Poisson distribution with parameter λ. For what values of λ is extinction certain? Write down expressions for the mean and variance of the number of individuals in generation n. Estimate the probability of extinction correct to one decimal place when $\lambda = 1.5$.

Answer
Suppose we have a population of individuals, each reproducing independently of the others, and that the probability distributions for the number of offspring of all individuals are identical. Let X_n denote the number of inderivdials in generation n. Then (X_n) is called a branching Markov chain.
The probability of extiniction when the probability has size 1 is given by the smallest positive root of $s = A(s)$. Since $A(s)$ is a power series with positive coefficients, it is concave upwards, and so its graph meets $y = s$ at most twice, for $s > 0$. Since $A(1) = 1$ for a p.g.f. this gives 3 possibilities

(i) PICTURE

(ii) PICTURE

(iii) PICTURE

So extinction happens with probability 1 if and only if $\mu \leq 1$

$$\begin{aligned} F_n(s) &= F_{n-1}(A(s)) \\ \text{So } F_n'(s) &= F_{n-1}'(A(s))A'(s) \end{aligned}$$

Putting s = 1 gives

$$\begin{aligned} F_n'(1) &= F_{n-1}'(1)A'(1) \text{ since A(1) = 1} \\ \mu_n &= \mu_{n-1} \cdot \mu \end{aligned}$$

Since $\mu_0 = 1$ we have $\mu_n = \mu^n$
Differentiating again gives

$$F_n''(s) = F_n''(1)(A(s))(A'(s)) + F_{n-1}'(1)A''(s)$$

Putting s=1 gives

$$F_n''(1) = f_{n-1}''(1)A'(1)^2 + F_{n-1}'(1)A''(1)$$

$$\begin{aligned} \sigma_n^2 + \mu_n^2 - \mu_n &= \mu^2(\sigma_{n-1}^2 + \mu_{n-1}^2 - \mu_{n-1}) + \mu_{n-1}(\sigma^2 + \mu^2 - \mu) \\ \sigma_n^2 + \mu^{2n} = \mu^n &= \mu^2(\sigma_{n-1}^2 + \mu^{2n-2} - \nu^{n-1}) + \mu^{n-1}(\sigma^2 + \mu^2 - \mu) \\ \sigma_n^2 &= \mu^2\sigma_{n-1}^2 + \mu^{n-1}\sigma^2 \\ &= \mu^2(\mu^2\sigma_{n-2}^2 + \mu^{n-2}\sigma^2) + \mu^{n-1}\sigma^2 \\ &= \mu^4\sigma_{n-2}^2 + \sigma(\mu^{n-1} + \mu^n) \\ &= \mu^6\sigma_{n-3}^2 + \sigma^2(\mu^{n-1} + \mu^n + \mu^{n+1} = ... \\ &= \mu^{2n-2}\sigma_1^2 + \sigma^2(\mu^{n-1} + ... + \mu^{2n-3}) \\ &= \sigma^2(\mu^{n-1} + \mu^n + .. + \mu^{2n-2} \\ &= \begin{cases} \sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right) & \mu \neq 0 \\ n\sigma^2 & \mu = 1 \end{cases} \end{aligned}$$

The mean and variance of the Poisson distribution are both λ
so $\mu_n = \lambda^n$ and $\sigma_n^2 = \begin{cases} \lambda^n\left(\frac{1-\lambda^n}{1-\lambda}\right) & \lambda \neq 0 \\ n & \lambda = 1 \end{cases}$
The p.g.f. for the poisson (λ) is $e^{\lambda(s-1)}$. So we have to estimate the solution for $e^{1.5(s-1)} = s$

	s		$e^{1.5(s-1)}$
Inetial guess	0.5	>	0.47...
	0.4	<	0.41...
	0.45	>	0.44...

Thus the extinction probability is 0.4 to 1 d.p.