Question

State Rouche's theorem and use it to show that all the roots of the equation

$$z^5 + \alpha z^2 + 1 = 0,$$

where the constant α satisfies $|\alpha| \leq 7$, lie inside the circle $|z| = 2$.
Assume that $\alpha = 1 + ib$ where $b > 1$. Use the argument principle to show that just two of these roots lie in the first quadrant.

Answer

Rouche's Theorem
Let $f(z) = z^5$ and $g(z) = \alpha z^2 + 1$
For $|z| = 2$ $|f(z)| = 32$, $|g(z)| \leq |\alpha||z|^2 + 1 \leq 7.4 + 1 = 29$ By Rouche's theorem f and $f + g$ have the same number of roots inside $|z| = 2$ i.e. all the 5 roots of $f + g$ are in C.

Suppose $\alpha = 1 + ib$ $b > 1$
DIAGRAM
The number of zeros $=\dfrac{1}{2\pi}[\arg g(z)]_C$
On the real axis $f(x) = x^5 + x^2 + 1 + ibx^2$
$\tan\arg g(z) = \dfrac{bx^2}{x^5 + x^2 + 1}$ this is continuous for $x \geq 0$, zero when $x = 0$ and $\to 0$ as $x \to \infty$. So the change in $\arg g(z)$ along OA is $< \epsilon_1$.
On the imaginary axis
$g(z) = (iy)^5 + (iy)^2 + 1 + ib(iy)^2 = iy^5 - iby^2 + y^2 + 1$
So $\tan\arg f(z) = \dfrac{y^5 - by^2}{y^2 + 1}$ this is continuous, zero at $y = 0$ and $\to \infty$ as $y \to \infty$. So the change in $\arg g(z)$ along BO is $-\frac{\pi}{2} + \epsilon_2$.
On the circle C, $z = Re^{i\theta}$
$g(z) = R^5 e^{5i\theta}\left(1 + \dfrac{e^{-3i\theta}}{R^2} + \dfrac{e^{-5i\theta}}{R^5} + \dfrac{ibe^{-2i\theta}}{R^2}\right)$
$\arg g(z) = 5\theta + arg(1 + w)$
w is small if R is large, so $\arg(1 + w)$ varies little,
so $[\arg g(z)]_{\text{arc } AB} = \dfrac{5\pi}{2} + \epsilon_3$
So $\dfrac{1}{2\pi}[\arg g(z)] = \dfrac{1}{2\pi}\left[\dfrac{5\pi}{2} - \dfrac{\pi}{2} + \epsilon\right] = 2$ as it is an integer
So $g(z)$ has 2 roots in the first quadrant.