Question

a) Use Cauchy's integral representation with an appropriate contour to show that, for $|z| < 1$,

$$z^n = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(n+1)\theta}}{e^{i\theta} - z} d\theta$$

where n is a positive integer. What is the value of the integral if $|z| > 1$?

b) Express the function

$$f(z) = \frac{2}{z(z-1)(z-2)}$$

in partial fractions.

Find the Laurent expansions of $f(z)$ in powers of z in each of the two regions $1 < |z| < 2$ and $|z| > 2$.

Hence, or otherwise, evaluate $\int_C f(z)dz$ where C is

i) the circle $|z| = \dfrac{3}{2}$

ii) the circle $|z| = 3$.

Answer

a) For $C =$ unit circle and $|z| < 1$, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(w)}{w - z} dw$

so with $f(z) = z^n$ and $z = e^{i\theta}$, $0 \le \theta \le 2\pi$,

$$z^n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{in\theta} i e^{i\theta} d\theta}{e^{i\theta} - z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(n+1)\theta} d\theta}{e^{i\theta} - z}$$

Now if $|z| > 1$, z is outside C and $\dfrac{f(w)}{w - z}$ is analytic inside and on C and so $\displaystyle\int_C \frac{w^n}{w - z} dw = 0$ by Cauchy's Theorem.

b) $f(z) = \dfrac{2}{z(z-1)(z-2)} = \dfrac{1}{z} - \dfrac{2}{z-1} + \dfrac{1}{z-2}$

For $1 < |z| < 2$, $\dfrac{1}{z-1} = \dfrac{1}{z}\left(1 - \dfrac{1}{z}\right)^{-1} = \dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \cdots\right)$

$\dfrac{1}{z-2} = -\dfrac{1}{2}\left(1 - \dfrac{z}{2}\right)^{-1} = -\dfrac{1}{2}\left(1 + \dfrac{z}{2} + \dfrac{z^2}{4} + \cdots\right)$

So $f(z) = \cdots - \dfrac{z^n}{2^{n+1}} - \cdots - \dfrac{z^2}{8} - \dfrac{z}{4} - \dfrac{1}{2} - \dfrac{1}{z} - \dfrac{2}{z^2} - \dfrac{2}{z^3} - \cdots$

For $|z| > 2$, $\dfrac{1}{z-1}$ expands as before, but

$\dfrac{1}{z-2} = \dfrac{1}{z}\left(1 - \dfrac{2}{z}\right)^{-1} = \dfrac{1}{z}\left(1 + \dfrac{2}{z} + \left(\dfrac{2}{z}\right)^2 + \cdots\right)$

So $f(z) = \dfrac{2}{z^3} + \dfrac{6}{z^4} + \dfrac{14}{2^5} + \cdots + \dfrac{2^{n-1} - 2}{z^n} + \cdots$

i) For the circle $C_1 : |z| = \frac{3}{2}$, this encloses $z = 0$ and $z = 1$

$$\int_{C_1} f(z)dz = \int_{C_1} \frac{1}{z} - 2\int_{C_1} \frac{2}{z-1} + \int_{C_1} \frac{1}{z-2}$$
$$= 2\pi i - 2.2\pi i + 0 = -2\pi i$$

ii) For the circle $C_2 : |z| = 3$, this encloses $z = 0$, $z = 1$ and $z = 2$

$$\int_{C_2} f(z)dz = 2\pi i - 2.2\pi i + 2\pi i = 0$$