QUESTION A random variable X has pdf $f(x) = (\beta+1)x^\beta \quad 0 \leq x \leq 1, \quad \beta < -1$. A random sample of n values of X, $x_1, x_2, ..., x_n$ has been obtained.

(i) Find $E(X)$ and use the method of moments to estimate β.

(ii) Show that the likelihood function L is given by

$$\ln L = n\ln(\beta+1) + \beta \sum \ln(x_i)$$

and hence find the maximum estimate of β.

ANSWER $f(x) = (\beta+1)x^\beta \quad 0 \leq x \leq 1$

(i) $E(X) = \int_0^1 (\beta+1)xx^\beta\, dx = [\frac{\beta+1}{\beta+2}x^{\beta+2}]_0^1 = \frac{\beta+1}{\beta+2}$
Method of moments $\overline{x} = \frac{\hat{\beta}+1}{\hat{\beta}+2} \quad \overline{x}(\hat{\beta}+2) = \hat{\beta}+1 \quad \hat{\beta} = \frac{2\overline{x}-1}{1-\overline{x}}$

(ii) $L = (\beta+1)^n \prod_{i=1}^n x_1^\beta$
$\ln L = n\ln(\beta+1) + \beta\sum_{i=1}^n \ln(x_i)$
$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta+1} + \sum_{i=1}^n \ln(x_i) = 0$ when $\hat{\beta} = \frac{-n}{\sum_{i=1}^n \ln(x_i)} - 1$(Check this is a maximum, $\frac{\partial^2 \ln L}{\partial \beta^2} < 0$