Question

Use the method of variation of parameters to find a particular solution to the following equations:

NOTE: you may wish to use earlier methods to check the answers but your solutions must use variation of parameters to get the solution.

1. $y'' - 5y' + 6y = 2e^x \qquad (*)$
2. $y'' + 2y' + y = 3e^{-x}$
3. $y'' + y = \tan x \quad 0 < x < \pi/2$
4. $y'' - 2y' + y = x^{3/2}\, e^x \qquad (*)$

Answer

1. Find y_c from $y_c'' - 5y_c' + 6y_c = 0$
   Auxiliary equation is $m^2 - 5m + 6 = 0$ with solutions $m = 2$, $m = 3$.
   Hence $y_c = Ae^{2x} + Be^{3x}$ and therefore take $y_1 = e^{2x}$, $y_2 = e^{3x}$
   Wronskian is
   $$W(x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x}$$
   $$v_1(x) = -\int \frac{y_2(x)\,(2e^x)}{W(x)}\,dx = -\int \frac{2e^{4x}}{e^{5x}}\,dx = -\int 2e^{-x}\,dx = 2e^{-x}$$
   $$v_2(x) = \int \frac{y_1(x)\,(2e^x)}{W(x)}\,dx = \int \frac{2e^{3x}}{e^{5x}}\,dx = \int 2e^{-2x}\,dx = -e^{-2x}$$
   $$y_{pi} = v_1y_1 + v_2y_2 = 2e^{-x}e^{2x} + \left(-e^{-2x}\right)e^{3x} = 2e^x - e^x = e^x$$
   General solution is $y = Ae^{2x} + Be^{3x} + e^x$

2. Find y_c from $y_c'' + 2y_c' + y_c = 0$
   Auxiliary equation is $m^2+2m+1 = 0$ with solutions $m = -1$ (repeated root).
   Hence $y_c = Ae^{-x} + Bxe^{-x}$ and therefore take $y_1 = e^{-x}$, $y_2 = xe^{-x}$
   Wronskian is
   $$W(x) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$

$v_1(x) = -\int \frac{y_2(x)\,(3e^{-x})}{W(x)}\ dx = -\int \frac{xe^{-x}3e^{-x}}{e^{-2x}}\ dx = -\int 3x\ dx = -\frac{3}{2}x^2$

$v_2(x) = \int \frac{y_1(x)\,(3e^{-x})}{W(x)}\ dx = \int \frac{e^{-x}3e^{-x}}{e^{-2x}}\ dx = \int 3\ dx = 3x$

$y_{pi} = v_1y_1 + v_2y_2 = -\frac{3}{2}x^2e^{-x} + 3x^2e^{-x} = \frac{3}{2}x^2e^{-x}$

General solution is $y = Ae^{-x} + Bxe^{-x} + \frac{3}{2}x^2e^{-x}$

3. Find y_c from $y_c'' + y_c = 0$
   Auxiliary equation is $m^2 + 1 = 0$ with solutions $m = i$, $m = -i$.
   Hence $y_c = A\sin x + B\cos x$ and therefore take $y_1 = \sin x$, $y_2 = \cos x$
   Wronskian is

$$W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

$v_1(x) = -\int \frac{y_2(x)\frac{\sin x}{\cos x}}{W(x)}\ dx = -\int \frac{\sin x}{-1}\ dx = -\cos x$

$v_2(x) = \int \frac{y_1(x)\frac{\sin x}{\cos x}}{W(x)}\ dx = \int \frac{\frac{\sin^2 x}{\cos x}}{-1}\ dx = \sin x - \ln(\sec x + \tan x)$

(Note: You can evaluate the integral as follows or other ways including using computer packages such as Maple)

$\left\{ \quad -\int \frac{\sin^2 x}{\cos x}\ dx = -\int \frac{1-\cos^2 x}{\cos x}\ dx = \int \cos x\ dx - \int \frac{1}{\cos x}\ dx = \sin x - \int \frac{\cos x}{\cos^2 x}\ dx = \sin x - \int \frac{\cos x}{1-\sin^2 x}\ dx = \sin x - \int \frac{\cos x}{2(1-\sin x)}\ dx + \int \frac{\cos x}{2(1+\sin x)}\ dx = \sin x + \frac{1}{2}\ln\left(\frac{1-\sin x}{1+\sin x}\right) = \sin x + \frac{1}{2}\ln\left(\frac{(1-\sin x)^2}{1-\sin^2 x}\right) \right\}$

$y_{pi} = v_1y_1 + v_2y_2 = -\sin x\cos x + \cos x\sin x - \cos x\ \ln(\sec x + \tan x)$

$y_{pi} = -\cos x\ \ln(\sec x + \tan x)$

General solution is $y = A\sin x + B\cos x - \cos x\ \ln(\sec x + \tan x)$

4. Find y_c from $y_c'' - 2y_c' + y_c = 0$

Auxiliary equation is $m^2 - 2m + 1 = 0$ with solutions $m = 1$ (repeated root).

Hence $y_c = Ae^x + Bxe^x$ and therefore take $y_1 = e^x$, $y_2 = xe^x$

Wronskian is

$$W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

$v_1(x) = -\int \frac{y_2(x)\left(x^{3/2}e^x\right)}{W(x)}\ dx = -\int \frac{xe^x x^{3/2} e^x}{e^{2x}}\ dx = -\int x^{5/2}\ dx = -\frac{2}{7}x^{7/2}$

$v_2(x) = \int \frac{y_1(x)\left(x^{3/2}e^x\right)}{W(x)}\ dx = \int \frac{e^x x^{3/2} e^x}{e^{2x}}\ dx = -\int x^{3/2}\ dx = \frac{2}{5}x^{5/2}$

$y_{pi} = v_1 y_1 + v_2 y_2 = -\frac{2}{7}x^{7/2}e^x + \frac{2}{5}x^{7/2}e^x = \frac{4}{35}x^{7/2}e^x$

General solution is $y = Ae^x + Bxe^x + \frac{4}{35}x^{7/2}e^x$