Question

(a) Suppose that events occur randomly in time, and that the time intervals between successive events are independent and identically distrusted, having a negative exponential distribution. Prove that such a sequence of events forms a Poisson process.

You may assume without proof that a sum of i.i.d. negative exponential random variables has a gamma distribution, $\Gamma(n, \lambda)$ with p.d.f.

$$\frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}$$

Light bulbs have an average lifetime of 200 days, and are replaced as soon as they fail. The time intervals between replacements are independent and identically distrusted, having a negative exponential distribution. What is the probability that at least 1000 days have elapsed since

(i) the last new light bulb was fitted,

(ii) the next to last light bulb was fitted?

(b) A machine needs two transistors to function, one of type A and one of type B. Both types fail independently according to a Posson processes, type B twice as often as type A on average. Gives that the machine has failed 10 times, what is that probability that 5 failures are due to type A and 5 are due to type B. Justify your conclusion.

Answer

(a) Let $N(t)$ denote the number of events which occur in time t. Let W_n denote the waiting time till the n-th event. $W_n \sim \Gamma(n, \lambda)$ for some λ.

So

$$\begin{aligned} P(N(t) = n) &= P(W_n \leq t) - P(W_{n+1} \leq t) \\ &= \int_0^t \frac{\lambda \cdot (\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} \, dx - \int_0^t \frac{\lambda (\lambda x)^n e^{-\lambda x}}{n!} \, dx \\ &= \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\ &\quad \text{(Integrating the 1st by parts)} \end{aligned}$$

This is the Poisson probability.

If light bulbs have an average lifetime of 200 days the they fail according to a poisson process of the rate $\lambda = \frac{1}{200}$. The number failing in 100 days has a poisson distribution with parameter 5.

(i) The required probability is

$$P(N = 0) = e^{-5} \approx 0.0067$$

(ii) The required probability is

$$P(N = 0) + P(N = 1) = e^{-5} + 5e^{-5} \approx 0.0404$$

(b) Suppose λ is the Poisson Parameter for type A failure. Then 2λ is the poisson parameter for type B failure. Let N_A and N_B denote the number of failures of each type in time t. Then

$$\begin{aligned}
P(N_A = 5|N_A + N_B + 10) &= \frac{P(N_A = 5 \text{ and } N_B = 10)}{P(N_A + N_B = 10)} \\
&= \frac{\frac{e^{-\lambda t}(\lambda t)^5}{5!} \frac{e^{-2\lambda t}(2\lambda t)^5}{5!}}{\frac{e^{-3\lambda t}(3\lambda t)^{10}}{10!}} \\
&= \frac{10!}{5!5!} \cdot \frac{2^5}{3^{10}} \\
&= \frac{252 \times 32}{59049} \\
&\approx 0.137
\end{aligned}$$