Question

Explain what a branching chain is. Suppose a population is descended from a single individual (generation 0). Let $A(s)$ denote the probability generating function for the number of offspring of any individual. Let $F_n(s)$ denote the probability generating function for the number of individuals in generation n.

Prove that

$$\begin{aligned} F_n(s) &= F_{n-1}(A(s)) \quad \text{and dedcue that} \\ F_n(s) &= A(F_{n-1}(s)). \end{aligned}$$

A game is played with coins as follows. Each coin has a probability $\frac{2}{3}$ of showing a head when tossed. Player A starts with one coin. This is given to player B, who takes it and tosses it until the first head is obtained. If the number of tosses needed <u>before</u> the first head is obtained is k, then k coins are given to player A. For the next stage of the game all the coins held by player A are given to player B, who takes each one and tosses it until a head is obtained, independently of the other coins. Again if the number of tosses needed before the first head shows for any particular coin is k, then k coins are given to player A, who therefore finishes this stage of the game with a certain number of coins. This process is repeated for all subsequent stages of the game.

Write down the probability that k tosses occur before the fist head. Use this to show that the probability generating function $A(s)$ for the number if tosses needed before the first heads is $\frac{2}{3-s}$. Use the relation deduced above to show, by induction or otherwise, that the probability generating function for the total number of coins held by player A after n stages of the game is given by

$$F_n(s) = \frac{2[2^n - 1 - (2^{n-1} - 1)s]}{2^{n+1} - 1 - (2^n - 1)s}$$

Show that this game terminates with probability 1.

Answer

Suppose we have a population of individuals, each reproducing independently of the others,Suppose the distributions of the number of offspring of all individuals are identical. Let X_n denote the number of individuals in generation n. Then (X_n) is a branching Markov chain.

Suppose $P(Z+k) = a_k$ and $A(s) = \sum_{k=0}^{\infty} a_k s^k$
Now $P(X_n = l | X_{n-1} = j) = P(Z_1 + ... + Z_j = l)$
$=$ coefficient of s^l in $[A(s)]$ as the Z_i's are i.i.d.
So

$$\begin{aligned} P(X_n = l) &= \sum_{j=0}^{\infty} P(X_n = l | X_{n-1} = j) \\ F_n(s) &= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (\text{coeff of } s^k \text{ in } [A(s)]^j) P(X_{n-1} = j) s^l \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} (\text{coeff of } s^k \text{ in } [A(s)]^j) s^l \right) P(X_{n-1} = j) \\ &= \sum_{j=0}^{\infty} P(X_{n-1} = j) [A(s)]^j \\ &= F_{n-1}(A(s)) \end{aligned}$$

Now $P(X_0 = 1) = 1$ so $F_0(s) = s$. Thus $F_1(s) = F_0(A(s)) = A(s)$
$F_2(s) = F_1(A(s)) = A(A(s))$
$F_n(s) = \underbrace{A(A(A(.....}_{\text{n times}}(s))...)) = A(F_{n-1}(s))$

The game is a branching Markov chain. Let X denote the number of tosses before the first head. Then we have
$P(X = k) = \left(\frac{1}{3}\right)^k \cdot \frac{2}{3} \quad k = 0, 1, ...$
So $A(s) = \sum \left(\frac{1}{3}\right)^k \cdot \frac{2}{3} s^k = \frac{2}{3} \frac{1}{1 - \frac{s}{3}} = \frac{2}{3 - s}$
Now for n=1 $F_1(s) = A(s)$ and the formula reduces to $\frac{2}{3-s}$.
Assume the formula is correct for n.
Then

$$\begin{aligned} F_{n+1}(s) &= A(F_n(s)) \\ &= \frac{2}{3 - F_n(s)} \\ &= \frac{2}{\left\{ 3 - \frac{2[2^n - 1 - (2^{n-1} - 1)s]}{2^{n+1} - 1 - (2^n - 1)s} \right\}} \\ &= \frac{2 \cdot [2^{n+1} - 1 - (2^n - 1)s]}{2^{n+2} - 1 - (2^{n+1} - 1)s} \end{aligned}$$

after apporx 3 lines of algebra

Hence the result, by induction.
The probability that the game terminates is the smallest positive root of $s = A(s)$. So $s = \dfrac{2}{3-s}$, i.e. $s^2 - 3s + 2 = 0$, giving $(s-1)(s-2) = 0$. s=1 is the smallest root, so the game terminates with probability 1.