QUESTION

(i) On a single set of axes sketch the graphs of $y = \cosh x$ and $y = \cosh^{-1} x$. State the domains and ranges of these functions.

(ii) If $y = \cosh^{-1} x$ use the exponential definition of $\cosh y$ to show that y satisfies the equation

$$e^{2y} - 2xe^{y} + 1 = 0.$$

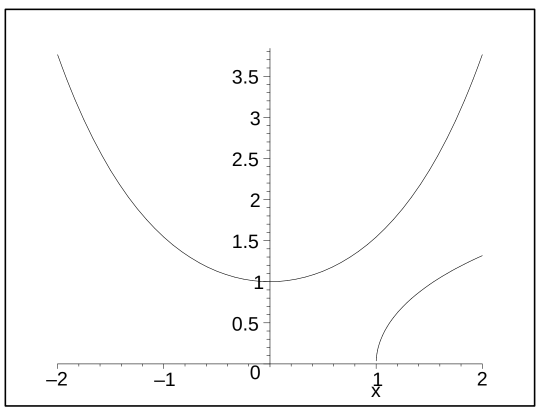
(iii) Rewrite the above equation as a quadratic and hence deduce that

$$\cosh^{-1} x = \ln\{x + \sqrt{x^2 - 1}\}.$$

(iv) By differentiating the result in (iii) verify that

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

ANSWER

(i) 

For cosh : the domain is $-\infty < x < \infty$ The range is $y \geq 1$.

For $\cosh^{-1}$: the domain is $x \geq 1$ The range is $y \geq 0$.

(ii) $y = \cosh^{-1} x$ therefore $x = \cosh y$

Thus $x = \dfrac{e^y + e^{-y}}{2}$ i.e. $2xe^y = e^{2y} + 1$ or $e^{2y} - 2xe^y + 1 = 0$

(iii) Hence $(e^y)^2 - 2xe^y + 1 = 0$ which is a quadratic in e^y so

$$
\begin{aligned}
e^y &= \frac{2x \pm \sqrt{((-2x)^2 - 4(1)(1))}}{2} \\
&= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\
e^y &= x \pm \sqrt{x^2 - 1} \\
\Rightarrow y &= \ln\{x \pm \sqrt{x^2 - 1}\}
\end{aligned}
$$

Now

$$
\begin{aligned}
&\ln\{x + \sqrt{x^2 - 1}\} + \ln\{x - \sqrt{x^2 - 1}\} \\
&= \ln\{(x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1})\} \\
&= \ln\{x^2 - (x^2 - 1)\} \\
&= \ln 1 = 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\ln\{x - \sqrt{x^2 - 1}\} &= -\ln\{x + \sqrt{x^2 - 1}\} \\
\text{so } y &= \pm \ln\{x + \sqrt{x^2 - 1}\}
\end{aligned}
$$

When $x \geq 1,\ x + \sqrt{x^2 + 1} \geq 1 \Rightarrow \ln\{x + \sqrt{x^2 - 1}\} \geq 0$
Therefore for the inverse, $\cosh^{-1} x = \ln\{x + \sqrt{x^2 - 1}\}$

(iv)

$$
\begin{aligned}
\frac{d}{dx}\left\{\cosh^{-1} x\right\} &= \frac{1}{x + \sqrt{x^2 - 1}} \times \left\{1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x)\right\} \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \left\{1 + \frac{x}{\sqrt{x^2 - 1}}\right\} \\
&= \frac{\sqrt{x^2 - 1} + x}{\left\{x + \sqrt{x^2 - 1}\right\} \sqrt{x^2 - 1}} \\
&= \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}
$$