QUESTION

A line L_1 passes through the two points $A(2,-4,1)$ and $B(5,2,-2)$ and a second line L_2 passes through the two points $C(2,-2,5)$ and $D(2,-1,7)$.

(i) Obtain the vector equations of the lines L_1 and L_2.

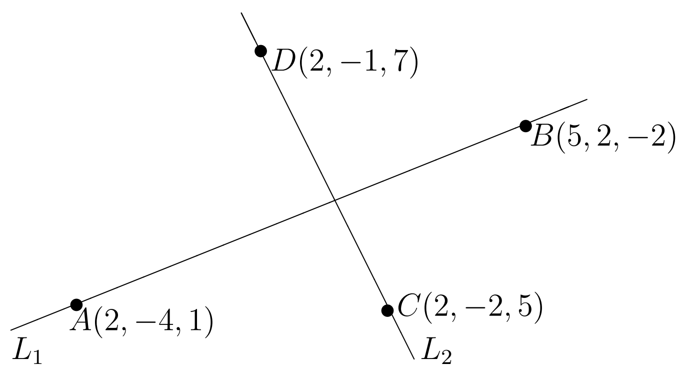
(ii) Find the point of intersection of L_1 and L_2.

(iii) Show that the lines L_1 and L_2 are perpendicular.

(iv) Find the area of the triangle ABD.

(v) Find the acute angle between the line L_2 and the line joining the points B and D.

ANSWER



(i) $\vec{AB} = (5-2, 2-(-4), -2-2) = (3,6,-3)$

Therefore the equation of L_1 is $\mathbf{r} = (2,-4,1) + s(3,6,-3)$

$\vec{CD} = (2-2, -1-(-2), 7-5) = (0,1,2)$

Therefore the equation of L_2 is $\mathbf{r} = (2,-2,5) + t(0,1,2)$

(ii) At the point of intersection

$$\begin{aligned} 2+3s &= 2, \Rightarrow s = 0 \\ -4+6s &= -2+t, \Rightarrow t = -2 \\ 1-3s &= 5+2t \end{aligned}$$

The third equation is satisfied by $s=0, t=-2$ therefore the point of intersection is $(2,-4,1) + 0(3,6,-3) = (2,-4,1)$, i.e. point A.

(iii) L_1 is parallel to $(3, 6, -3)$, L_2 is parallel to $(0, 1, 2)$.

$(3, 6, -3).(0, 1, 2) = 0 + 6 - 6 = 0$ therefore the lines are perpendicular.

(iv) Since L_1 and L_2 intersect at A, area $ABD = \frac{1}{2}(AB)(AD)$
But
$AB = \{(5-2)^2 + (2-(-4))^2 + (-2-1)^2\}^{\frac{1}{2}} = \{9 + 36 + 9\}^{\frac{1}{2}} = \sqrt{54}$
$AB = \{(2-2)^2 + (-1-(-4))^2 + (7-1)^2\}^{\frac{1}{2}} = \{0 + 9 + 36\}^{\frac{1}{2}} = \sqrt{45}$
Therefore the area is $\frac{1}{2}\sqrt{54}\sqrt{45} = \frac{1}{2}3\sqrt{6}\ 3\sqrt{5} = \frac{9}{2}\sqrt{30} \sim 24.65$

(v) $\vec{BD} = (2-5, -1-2, 7-(-2)) = (-3, -3, 9) = 3(-1, -1, 3)$
L_2 is parallel to $(0, 1, 2)$ therefore

$$\cos\theta = \frac{(-1, -1, 3).(0, 1, 2)}{\sqrt{((-1)^2 + (-1)^2 + 3^2)}\sqrt{(1^2 + 3^2)}} = \frac{0 - 1 + 6}{\sqrt{11}\sqrt{5}} = \frac{5}{\sqrt{55}}$$

$\theta = 47.6^\circ (= 0.831 \text{ radians})$