Question

Suppose that the proportion of defective items in a large manufactured lot is 0.1. What is the smallest random sample of items that must be taken from the lot in order for the probability to be at least 0.99 that the proportion of defective items in the sample will be less than 0.13?

Answer

Let $X_i = \left\{ \begin{array}{ll} 1 & \text{if the ith chosen item is defective} \\ 0 & \text{otherwise} \end{array} \right\}$

Now $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ = sample proportion of defective items

The problem is what minimum n we require so that

$$P(\bar{X}_n < 0.13) \geq 0.99$$

Here $E(X_i) = \mu = 0.1$ and $\text{var}(X_i) = (0.1)(1-0.1) = 0.09 = \sigma^2$ since X_i are Bernoulli random variables.

CLT

$Z = \dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$ approximately when n is large.

Therefore

$$\begin{aligned} & P(\bar{X}_n < 0.13) \\ = \; & P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < \frac{(0.13-0.1)\sqrt{n}}{\sqrt{0.09}}\right) \\ = \; & P\left(Z < \frac{0.03}{0.03}\sqrt{n}\right) \text{ where } Z \sim N(0,1) \\ = \; & P\left(Z < \frac{\sqrt{n}}{10}\right) \end{aligned}$$

i.e. $\Phi\left(\frac{\sqrt{n}}{10}\right) \geq \Phi(2.326) = 0.99$

$\Rightarrow \frac{n}{100} \geq 2.326^2$ i.e. $n \geq 541.02$

Therefore the smallest n we require is <u>542</u>.