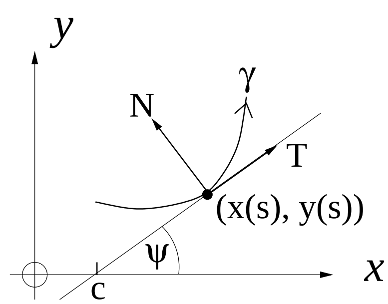
Question

Verify that if the tangent to a plane curve γ at the point $\gamma(s)$ makes an angle $\psi(s)$ with the x-axis (turning from the positive x-direction to the tangent vector) then $\kappa(s) = \frac{d\psi}{ds}(s)$ (where $s = \text{arclength}$ as usual). Suppose $y'(s) \neq 0$ and let the tangent line at $\gamma(s)$ meet the x-axis at $x = c(s)$, say. Show that if $y(s) \neq 0$ (so the point $\gamma(s)$ is not on the x-axis itself) then $\kappa(s) = 0$ if and only if $c'(s) = 0$.

Answer

Very easy: we see that

$$\begin{aligned} T(s) &= (\cos\phi, \sin\phi) \\ \Rightarrow T'(s) &= (-\sin\phi, \cos\phi) \\ \textbf{But } N(s) &= (-\sin\phi, \cos\phi) \\ \Rightarrow K(s) &= \frac{d\phi}{ds}. \end{aligned}$$



$\gamma(s) = (x(s), y(s))$**, unit speed.**

Equation of tangent to curve γ at $(x(s), y(s))$ is

$$(y - y(s))x'(s) = (x - x(s))y'(s).$$

This meets the x-axis at $(c(s), 0)$ so we have

$$-y(s)x'(s) = (c(s) - x(s))y'(s). \longleftarrow (1)$$

Differentiate: $-y'x' = yx'' = (c' - x')y' + (c - x)y''$**, (dropping the s).**

Giving:

$$-yx'' = c'y' + (c - x)y''$$

Substitute $(c - x)$ from (1) to get (with $y' \neq 0$)

$$\begin{aligned} -yx'' &= c'y' + \frac{-yx'}{y'}.y'' \\ \textbf{i.e. } c'(y')^2 &= y(x'y'' - y'x'') = yK \\ \textbf{as } x'(s)^2 + y'(s)^2 &= 1 \end{aligned}$$

Hence since $y \neq 0$ we see $K = 0$ precisely when $c' = 0$.